

Vectors, Polar Coordinates, and Complex Numbers: math lesson, Grade 12 – download the PDF

Pilots, game designers, and electrical engineers all rely on the same three ideas. A **vector** packages a size and a direction into one object. **Polar coordinates** locate a point by how far it is and which way to look. **Complex numbers** in polar form turn rotation and scaling into simple multiplication. In this chapter you will connect all three and use them to solve real problems.

1. Vectors: magnitude and direction

D Definition: vector and magnitude

A vector in the plane is written in component form as $\mathbf{v} = \langle a, b \rangle$. Its **magnitude** (length) is $\|\mathbf{v}\| = \sqrt{a^2 + b^2}$. Its **direction angle** θ is measured counterclockwise from the positive x -axis and satisfies $\tan \theta = \frac{b}{a}$ when $a \neq 0$.

Conversely, a vector with magnitude m and direction angle θ has components $\langle m \cos \theta, m \sin \theta \rangle$. The calculator's inverse tangent only returns angles between -90° and 90° , so you must always check which quadrant the vector points into.

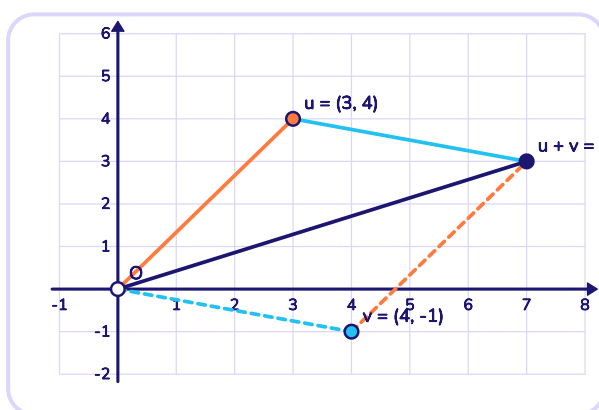
E Example 1: magnitude and direction

Find the magnitude and direction angle of $\mathbf{u} = \langle -6, 6\sqrt{3} \rangle$.

$\|\mathbf{u}\| = \sqrt{36 + 108} = \sqrt{144} = 12$. Then $\tan \theta = \frac{6\sqrt{3}}{-6} = -\sqrt{3}$, so the reference angle is 60° . The vector points left and up (quadrant II), so $\theta = 180^\circ - 60^\circ = 120^\circ$.
Answer: magnitude 12, direction 120° .

2. Vector operations

Vectors are added and subtracted component by component, and multiplied by a number (a *scalar*) the same way: $\langle a, b \rangle + \langle c, d \rangle = \langle a + c, b + d \rangle$ and $k\langle a, b \rangle = \langle ka, kb \rangle$. Geometrically, you add two vectors by placing them tip to tail; the sum is the diagonal of the parallelogram they form. Multiplying by k stretches the vector by the factor $|k|$ and reverses it when $k < 0$.



P Unit vectors

A **unit vector** has magnitude 1. To get the unit vector in the direction of $\mathbf{v}$, divide by the magnitude: $\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$. The vectors $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$ let you write $\langle a, b \rangle = a\mathbf{i} + b\mathbf{j}$.

E Example 2: a linear combination

Let $\mathbf{u} = \langle 2, 5 \rangle$ and $\mathbf{v} = \langle -3, 1 \rangle$. Then $2\mathbf{u} - 3\mathbf{v} = \langle 4, 10 \rangle - \langle -9, 3 \rangle = \langle 13, 7 \rangle$. The unit vector along $\langle 3, 4 \rangle$ is $\frac{1}{5}\langle 3, 4 \rangle = \langle 0.6, 0.8 \rangle$.

3. The dot product

D Definition: dot product

For $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$, where θ is the angle between the vectors ($0^\circ \leq \theta \leq 180^\circ$).

P Consequences

$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|}$. Two nonzero vectors are **perpendicular** exactly when $\mathbf{a} \cdot \mathbf{b} = 0$.

The projection of $\mathbf{a}$ onto $\mathbf{b}$ is $\text{proj}_{\mathbf{b}} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \mathbf{b}$. In physics, the work done by a constant force $\mathbf{F}$ over a displacement $\mathbf{d}$ is $W = \mathbf{F} \cdot \mathbf{d}$.

E Example 3: the angle between two vectors

With $\mathbf{u} = \langle 2, 5 \rangle$ and $\mathbf{v} = \langle -3, 1 \rangle$: $\mathbf{u} \cdot \mathbf{v} = -6 + 5 = -1$, $\|\mathbf{u}\| = \sqrt{29}$, $\|\mathbf{v}\| = \sqrt{10}$. So $\cos \theta = \frac{-1}{\sqrt{290}} \approx -0.0587$ and $\theta \approx 93.4^\circ$: the vectors are almost, but not quite, perpendicular.



Watch out

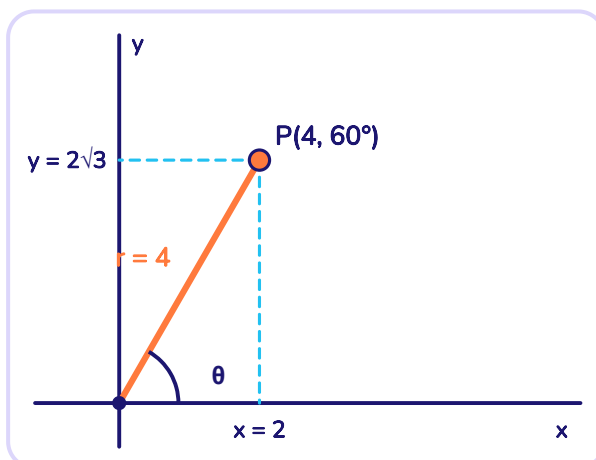
The dot product of two vectors is a *number*, not a vector. Also, $\|\mathbf{a} + \mathbf{b}\|$ is usually smaller than $\|\mathbf{a}\| + \|\mathbf{b}\|$; the two are equal only when the vectors point the same way.

4. Polar coordinates and conversions



Definition: polar coordinates

A point P is described by (r, θ) : r is the signed distance from the origin (the *pole*) and θ is the angle from the polar axis, the positive x -axis. A negative r means you walk backward, so $(-r, \theta)$ and $(r, \theta + \pi)$ name the same point.



Conversion formulas

Polar to rectangular: $x = r \cos \theta$, $y = r \sin \theta$. Rectangular to polar: $r^2 = x^2 + y^2$ and $\tan \theta = \frac{y}{x}$, with θ chosen in the correct quadrant.

Unlike rectangular coordinates, a point has infinitely many polar names: (r, θ) , $(r, \theta + 2\pi)$, $(-r, \theta + \pi)$, and so on. The pole itself is $(0, \theta)$ for any θ .

E

Example 4: converting in both directions

(a) $\left(6, \frac{5\pi}{6}\right)$: $x = 6 \cos 150^\circ = -3\sqrt{3}$ and $y = 6 \sin 150^\circ = 3$, so the point is $(-3\sqrt{3}, 3)$.

(b) $(-5, 5)$: $r = \sqrt{25 + 25} = 5\sqrt{2}$. The point is in quadrant II, so $\theta = \frac{3\pi}{4}$. Answer: $\left(5\sqrt{2}, \frac{3\pi}{4}\right)$.

5. Graphing polar curves

To graph $r = f(\theta)$, build a table of values, plot each pair (r, θ) , and connect the points in order of increasing θ . Symmetry saves work: if replacing θ by $-\theta$ leaves the equation unchanged, the graph is symmetric about the polar axis (the x -axis).

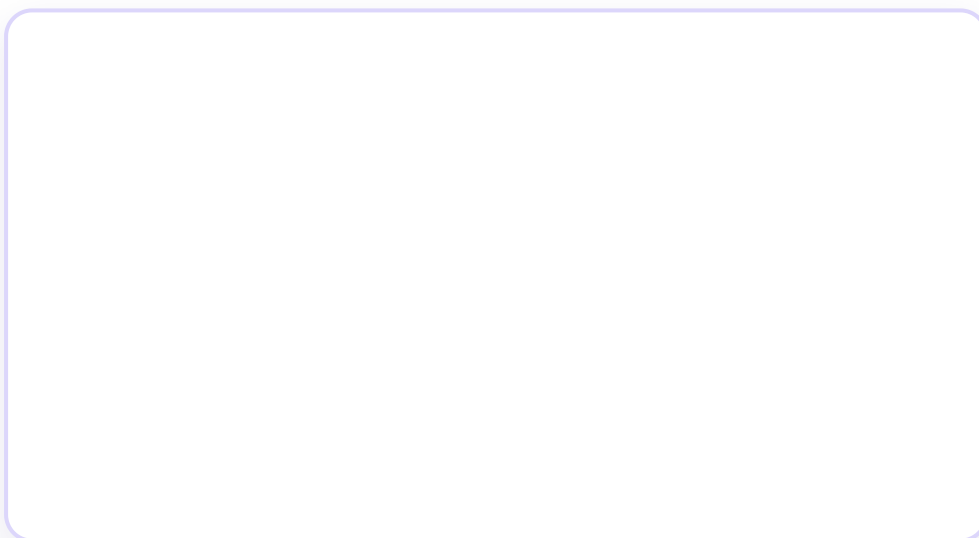
Equation	Graph	Key facts
$r = a$	Circle	Centered at the pole, radius $ a $
$\theta = \alpha$	Line	Through the pole at angle α
$r = 2a \cos \theta$	Circle	Center $(a, 0)$, radius $ a $
$r = a \pm a \cos \theta$	Cardioid	Heart shape, symmetric about the x -axis
$r = a \cos n\theta$ or $a \sin n\theta$	Rose	n petals if n is odd, $2n$ petals if n is even

E Example 5: a cardioid

Graph $r = 2 + 2 \cos \theta$ using a table.

θ	0	$\pi/3$	$\pi/2$	$2\pi/3$	π
$\cos \theta$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1
r	4	3	2	1	0

The curve starts 4 units to the right, shrinks to the pole at $\theta = \pi$, and its lower half mirrors the upper half. It is the left graph below; the right graph is the three-petal rose $r = 4 \cos 3\theta$.



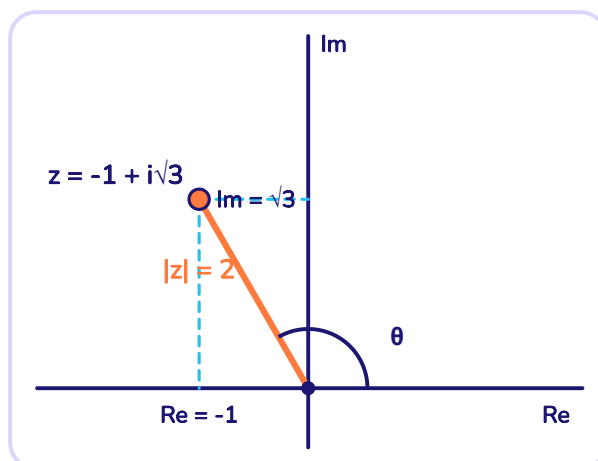
Vectors polar grade 12: a cardioid $r = 2 + 2 \cos \theta$ on the left and a three-petal rose $r = 4 \cos 3\theta$ on the right, both drawn on polar axes

6. Complex numbers in polar form

A complex number $z = a + bi$ is the point (a, b) of the complex plane (real axis horizontal, imaginary axis vertical). Its **modulus** is $|z| = r = \sqrt{a^2 + b^2}$ and its **argument** θ is the angle of the point, so $a = r \cos \theta$ and $b = r \sin \theta$.

D Definition: polar (trigonometric) form

$z = r(\cos \theta + i \sin \theta)$, often abbreviated $r \operatorname{cis} \theta$.



P Products and quotients

If $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$, then $z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$ and $\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$. Multiply the moduli, add the arguments.

E Example 6: polar form and a product

(a) For $z = -1 + i\sqrt{3}$: $r = \sqrt{1 + 3} = 2$, the point is in quadrant II with reference angle 60° , so $z = 2 \operatorname{cis} 120^\circ$.

(b) $(3 \operatorname{cis} 40^\circ)(2 \operatorname{cis} 50^\circ) = 6 \operatorname{cis} 90^\circ = 6i$.

7. De Moivre's Theorem

P De Moivre's Theorem

For every integer n , $[r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta)$. Raising to a power means raising the modulus to that power *and* multiplying the argument by n .

E Example 7: big powers made easy

(a) $1 + i = \sqrt{2} \operatorname{cis} 45^\circ$, so $(1 + i)^8 = (\sqrt{2})^8 \operatorname{cis} 360^\circ = 16$.

(b) $\sqrt{3} - i = 2 \operatorname{cis}(-30^\circ)$, so $(\sqrt{3} - i)^6 = 2^6 \operatorname{cis}(-180^\circ) = -64$.

⚠ Watch out

A frequent slip is to multiply the argument by n but forget to raise the modulus to the power n . Also reduce the final angle by multiples of 360° before reading off the answer.

8. Roots of complex numbers

P The n th roots formula

A nonzero number $z = r \operatorname{cis} \theta$ has exactly n distinct n th roots:

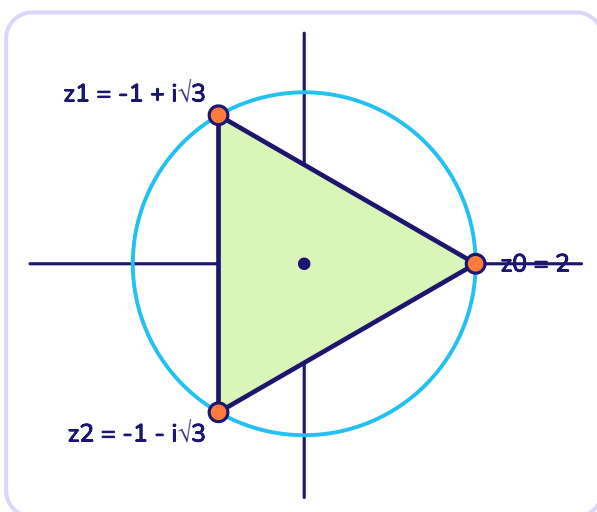
$z_k = r^{1/n} \operatorname{cis} \left(\frac{\theta + 360^\circ k}{n} \right)$ for $k = 0, 1, \dots, n - 1$. They lie on a circle of radius $r^{1/n}$ and are the vertices of a regular n -gon.

M**Method: find all n th roots**

1. Write z in polar form $r \operatorname{cis} \theta$.
2. Compute $r^{1/n}$ and the first argument θ/n .
3. Add $360^\circ/n$ repeatedly to get the other arguments.
4. Convert to $a + bi$ if the problem asks for it.

E**Example 8: cube roots of 8**

$8 = 8 \operatorname{cis} 0^\circ$, so the roots have modulus $\sqrt[3]{8} = 2$ and arguments $0^\circ, 120^\circ, 240^\circ$:
 $z_0 = 2, z_1 = -1 + i\sqrt{3}, z_2 = -1 - i\sqrt{3}$.

**!****Zyro's tip**

On my home planet we check roots by symmetry: the n roots are spread evenly around a circle, so they always add up to zero. If your sum is not zero, one root is wrong!

Key takeaways

- $\|\langle \mathbf{a}, \mathbf{b} \rangle\| = \sqrt{a^2 + b^2}$; the direction angle needs a quadrant check.
- Add vectors and multiply by scalars component by component; the dot product $a_1b_1 + a_2b_2$ is a number equal to $\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$.
- Perpendicular vectors have dot product 0; work is $W = \mathbf{F} \cdot \mathbf{d}$.
- $x = r \cos \theta, y = r \sin \theta, r^2 = x^2 + y^2$; a point has many polar names.
- Rose $r = a \cos n\theta$: n petals if n is odd, $2n$ if n is even; cardioid $r = a \pm a \cos \theta$.
- $z = r \operatorname{cis} \theta$: multiply moduli and add arguments; De Moivre gives $z^n = r^n \operatorname{cis} n\theta$.
- The n n th roots of z are equally spaced on a circle of radius $r^{1/n}$.



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