

# Vector Spaces and Linear Transformations: practice solutions, College – download the PDF

Written solutions to the chapter problems. Check each step, then correct yourself.

## 1 A line through the origin



The zero vector satisfies  $\mathbf{0} = 3 \cdot \mathbf{0}$ , so  $\mathbf{0} \in W$ . If  $(x_1, 3x_1)$  and  $(x_2, 3x_2)$  are in  $W$ , their sum is  $(x_1 + x_2, 3(x_1 + x_2))$ , which is in  $W$ . For a scalar  $c$ ,  $c(x, 3x) = (cx, 3cx) \in W$ . All three conditions hold, so  $W$  is a subspace. In fact  $W = \text{span}\{(1, 3)\}$ , a line of dimension 1.

## 2 A half-plane



The vector  $(1, 0)$  belongs to  $H$ , but  $(-1) \cdot (1, 0) = (-1, 0)$  has a negative first coordinate, so it is not in  $H$ . Closure under scalar multiplication fails, hence  $H$  is not a subspace (even though it contains  $\mathbf{0}$  and is closed under addition).

## 3 Two pairs of vectors



(a) Since  $(2, 4) = 2(1, 2)$ , the pair is dependent.

(b) Neither is a multiple of the other. Equivalently,  $\det \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} = 1 - 6 = -5 \neq 0$ . The pair is independent, hence a basis of  $\mathbb{R}^2$ .

#### 4 Counting dimensions



(a) 4. (b) 3, with basis  $1, x, x^2$ . (c)  $2 \cdot 3 = 6$ , one basis matrix per entry. (d) 3, with the three matrices that have a single 1 on the diagonal.

#### 5 Is it linear?



(a)  $T(x, y) = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$  is a matrix product, so  $T$  is linear.

$$T(1, 2) = (2 + 2, 1 - 2) = (4, -1).$$

(b)  $S(0, 0) = (1, 0) \neq (0, 0)$ , but every linear map sends  $\mathbf{0}$  to  $\mathbf{0}$ . So  $S$  is not linear.

#### 6 Rank-nullity quick check



The matrix has  $n = 6$  columns, so nullity  $= 6 - 3 = 3$ . The columns have 4 entries, so the column space is a 3-dimensional subspace of  $\mathbb{R}^4$ .

#### 7 Coordinates in a basis



Solve  $a(1, 0) + b(1, 1) = (4, 3)$ :  $a + b = 4$  and  $b = 3$ , so  $a = 1$ . Thus  $[v]_B = (1, 3)$ .

Check:  $(1, 0) + 3(1, 1) = (4, 3)$ .

#### 8 Basis of a solution set



From  $x - z = 0$  we get  $z = x$ ; then  $y = -x - z = -2x$ . So  $(x, y, z) = x(1, -2, 1)$ . The vector  $(1, -2, 1)$  is nonzero and spans  $W$ , so it is a basis and  $\dim W = 1$ . Check:

$$1 - 2 + 1 = 0 \text{ and } 1 - 1 = 0.$$

## 9 An affine plane



First way:  $(0, 0, 0)$  gives  $0 \neq 1$ , so the zero vector is missing. Second way:  $(1, 0, 0)$  and  $(0, 1, 0)$  are in the set, but their sum  $(1, 1, 0)$  has coordinate sum 2, so closure under addition fails.

## 10 Polynomial basis



Suppose  $ap_1 + bp_2 + cp_3 = 0$ . Comparing coefficients of  $1, x, x^2$  gives  $a + c = 0$ ,  $a + b = 0$ ,  $b + c = 0$ . Adding the three equations gives  $a + b + c = 0$ , so  $b = 0$ ,  $c = 0$ ,  $a = 0$ . The three polynomials are independent. Since  $\dim P_2 = 3$ , three independent vectors form a basis.

## 11 In the span or not?



We have  $au + cw = (a, a + c, a + 2c)$ .

(a)  $a = 3$ ,  $a + c = 5$  gives  $c = 2$ , and then  $a + 2c = 7$  matches. So  $(3, 5, 7) = 3u + 2w$ : yes.

(b) The same first two equations force  $a = 3$ ,  $c = 2$ , but then  $a + 2c = 7 \neq 8$ . The system is inconsistent: no.

## 12 Null and column space



Subtracting rows 1 and 2 from row 3 gives a zero row, so the echelon form is

$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ . Pivots are in columns 1 and 2, so the rank is 2 and  $\text{Col}(B)$  has basis

$(1, 0, 1), (0, 1, 1)$ . The free variable is  $x_3 = t$ ; then  $x_1 = -2t$  and  $x_2 = t$ , so  $\text{Nul}(B)$  has basis  $(-2, 1, 1)$ . Check:  $2 + 1 = 3$  columns.

### 13 A rotation



$T(e_1) = (0, 1)$  and  $T(e_2) = (-1, 0)$ , so the matrix is  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . Then  $T(3, 5) = (-5, 3)$  and  $T(-5, 3) = (-3, -5)$ . Two quarter turns make a half turn, and indeed  $(-3, -5) = -(3, 5)$ .

### 14 Kernel and range



$T(x, y, z) = \mathbf{0}$  means  $x = y$  and  $y = z$ , so  $\ker T = \{t(1, 1, 1)\}$  with basis  $(1, 1, 1)$  and nullity 1. By rank-nullity the rank is  $3 - 1 = 2 = \dim \mathbb{R}^2$ , so  $T$  is onto. It is not one-to-one because the kernel is not  $\{\mathbf{0}\}$ .

### 15 Trace-zero matrices



The zero matrix has trace 0. The trace is additive and  $\text{tr}(cM) = c \text{tr}(M)$ , so  $W$  is closed under both operations:  $W$  is a subspace. A matrix  $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$  in  $W$  equals  $a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ . These three matrices span  $W$  and are independent (compare entries), so  $\dim W = 3$ , one less than  $\dim M_{2 \times 2} = 4$ .

### 16 The derivative as a linear map



Linearity:  $(p + q)' = p' + q'$  and  $(cp)' = cp'$ . The kernel consists of polynomials with zero derivative, the constants:  $\ker D = \text{span}\{1\}$ , of dimension 1. Since  $\dim P_3 = 4$ , the rank is  $4 - 1 = 3 = \dim P_2$ , so the range is all of  $P_2$  and  $D$  is onto. Directly:  $ax^2 + bx + c = D\left(\frac{a}{3}x^3 + \frac{b}{2}x^2 + cx\right)$ .

### 17 Coordinates in a new basis

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$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ ,  $\det P = -2$ , so  $P^{-1} = \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$ . Then  $[v]_B = P^{-1}(7, 3) = (5, 2)$ . Check:  $5(1, 1) + 2(1, -1) = (7, 3)$ .

### 18 Matrix in a basis

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$T(1, 1) = (3, 5) = 4(1, 1) - 1(1, -1)$ , so the first column of  $[T]_B$  is  $(4, -1)$ .  
 $T(1, -1) = (-1, 1) = 0(1, 1) - 1(1, -1)$ , so the second column is  $(0, -1)$ . Therefore  $[T]_B = \begin{pmatrix} 4 & 0 \\ -1 & -1 \end{pmatrix}$ . Trace:  $4 - 1 = 3 = 1 + 2$ . Determinant:  $-4 - 0 = -4 = 1 \cdot 2 - 2 \cdot 3$ . Both match.

### 19 Preserving independence

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Suppose  $a(v_1 + v_2) + b(v_2 + v_3) + c(v_1 + v_3) = \mathbf{0}$ . Regrouping gives  $(a + c)v_1 + (a + b)v_2 + (b + c)v_3 = \mathbf{0}$ . By independence,  $a + c = 0$ ,  $a + b = 0$ ,  $b + c = 0$ . Adding gives  $2(a + b + c) = 0$ , so  $a + b + c = 0$ , and subtracting each equation yields  $b = 0$ ,  $c = 0$ ,  $a = 0$ . The new family is independent.

### 20 A parameter and dependence

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The vectors are dependent exactly when the determinant is zero:

$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & a & 1 \end{vmatrix} = 1(2 - a) - 1(1 - 2) + 0 = 3 - a$ . So  $a = 3$ . Then  $(2, 3, 1) = (1, 1, 0) + (1, 2, 1)$ , the relation  $v_1 + v_2 - v_3 = \mathbf{0}$ .

## 21 Recovering a map from a basis



Since  $e_1 = \frac{1}{2}[(1, 1) + (1, -1)]$ ,  $T(e_1) = \frac{1}{2}[(3, 1) + (1, 5)] = (2, 3)$ . Since  $e_2 = \frac{1}{2}[(1, 1) - (1, -1)]$ ,  $T(e_2) = \frac{1}{2}[(3, 1) - (1, 5)] = (1, -2)$ . So  $A = \begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix}$ . Then  $T(4, -2) = (8 - 2, 12 + 4) = (6, 16)$ . Check:  $A(1, 1) = (3, 1)$  and  $A(1, -1) = (1, 5)$ .



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