

Trigonometric Functions and the Unit Circle: math lesson, Grade 11 – download the PDF

Ocean tides, sound waves, the swing of a pendulum and the height of a seat on a Ferris wheel all repeat in a regular pattern. Trigonometric functions are the mathematical tool for describing that repetition. In this chapter you will measure angles in radians, learn to read the unit circle, graph sine, cosine and tangent, solve trigonometric equations and build your own models of periodic phenomena.

1. Radian measure and converting between degrees and radians

Degrees cut a full turn into 360 equal pieces, a choice that comes from history rather than from mathematics. A **radian** lets the circle itself set the unit.

D Radian

One radian is the measure of a central angle that cuts off an arc whose length equals the radius of the circle. Because the circumference is $2\pi r$, a full turn measures

$$\frac{2\pi r}{r} = 2\pi \text{ radians.}$$

So $360^\circ = 2\pi$ radians and $180^\circ = \pi$ radians. This single fact gives both conversion rules. When the angle θ is in radians, the arc length is simply $s = r\theta$.

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Conversion rules

Degrees to radians: multiply by $\frac{\pi}{180}$. Radians to degrees: multiply by $\frac{180}{\pi}$.

Degrees	Radians	Degrees	Radians
0°	0	135°	$\frac{3\pi}{4}$
30°	$\frac{\pi}{6}$	150°	$\frac{5\pi}{6}$
45°	$\frac{\pi}{4}$	180°	π
60°	$\frac{\pi}{3}$	270°	$\frac{3\pi}{2}$
90°	$\frac{\pi}{2}$	360°	2π
120°	$\frac{2\pi}{3}$	-90°	$-\frac{\pi}{2}$

E**Example 1: converting and finding an arc length**

(a) $135^\circ = 135 \times \frac{\pi}{180} = \frac{3\pi}{4}$ radians.

(b) $\frac{7\pi}{6}$ radians $= \frac{7\pi}{6} \times \frac{180}{\pi} = 210^\circ$.

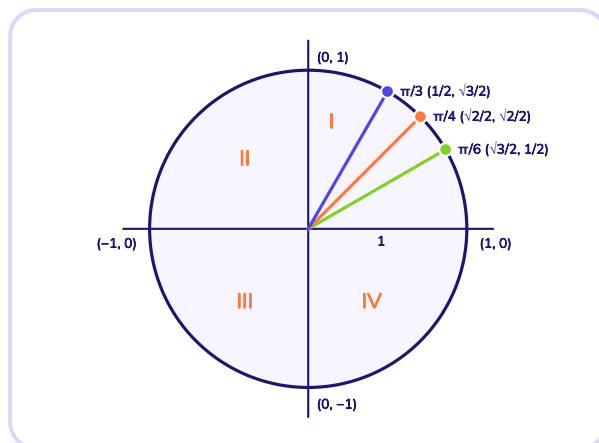
(c) A bicycle wheel of radius 14 inches turns through $\frac{2\pi}{3}$ radians. The arc length is $s = 14 \times \frac{2\pi}{3} = \frac{28\pi}{3} \approx 29.3$ inches, which is about 74.5 centimeters.

**Common mistake**

The formula $s = r\theta$ works only when θ is in radians. If an angle is given in degrees, convert it first.

2. The unit circle and exact values of sine, cosine and tangent**D****Unit circle**

The unit circle is the circle of radius 1 centered at the origin. Start at the point $(1, 0)$ and move counterclockwise along the circle through a signed angle θ . The point where you stop is $(\cos \theta, \sin \theta)$. The tangent is $\tan \theta = \frac{\sin \theta}{\cos \theta}$, defined whenever $\cos \theta \neq 0$.



The coordinates of the first-quadrant points come from two special triangles: the 45° - 45° - 90° triangle and the 30° - 60° - 90° triangle. They give the values you should know by heart.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

! Zyro's memory trick

For the sine of $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$, count the numbers $\frac{\sqrt{0}}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$.
Cosine uses the same list in reverse order.

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Example 2: reading the unit circle

The point for $\theta = \frac{\pi}{3}$ is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, so $\cos \frac{\pi}{3} = \frac{1}{2}$ and $\tan \frac{\pi}{3} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$. At $\theta = \pi$ the point is $(-1, 0)$, so $\sin \pi = 0$ and $\cos \pi = -1$.

3. Reference angles and signs by quadrant

Every angle can be handled with the first-quadrant values. The **reference angle** of θ is the acute angle between the terminal side of θ and the x-axis. Angles that differ by a multiple of 2π end at the same point, so they have the same trigonometric values.

Quadrant	Angle θ	$\sin \theta$	$\cos \theta$	$\tan \theta$	Reference angle
I	$0 < \theta < \frac{\pi}{2}$	+	+	+	θ
II	$\frac{\pi}{2} < \theta < \pi$	+	-	-	$\pi - \theta$
III	$\pi < \theta < \frac{3\pi}{2}$	-	-	+	$\theta - \pi$
IV	$\frac{3\pi}{2} < \theta < 2\pi$	-	+	-	$2\pi - \theta$

M**Method: exact value of any special angle**

1. Find the quadrant of the terminal side (subtract 2π if the angle is larger than a full turn).
2. Compute the reference angle with the table above.
3. Write the sine, cosine or tangent of the reference angle.
4. Attach the sign that the quadrant gives to that function.

E**Example 3: using a reference angle**

Find $\cos \frac{7\pi}{6}$ and $\tan \frac{5\pi}{3}$.

$\frac{7\pi}{6}$ lies in quadrant III, its reference angle is $\frac{7\pi}{6} - \pi = \frac{\pi}{6}$, and cosine is negative there. So $\cos \frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$.

$\frac{5\pi}{3}$ lies in quadrant IV, the reference angle is $2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$, and tangent is negative there. So $\tan \frac{5\pi}{3} = -\sqrt{3}$.

4. The Pythagorean identity

Every point (x, y) of the unit circle satisfies $x^2 + y^2 = 1$. Since $x = \cos \theta$ and $y = \sin \theta$, we get the fundamental identity.

P**Pythagorean identity**

For every real number θ : $\sin^2 \theta + \cos^2 \theta = 1$.

It lets you find one function when you know another and the quadrant. The quadrant decides which sign of the square root to keep.

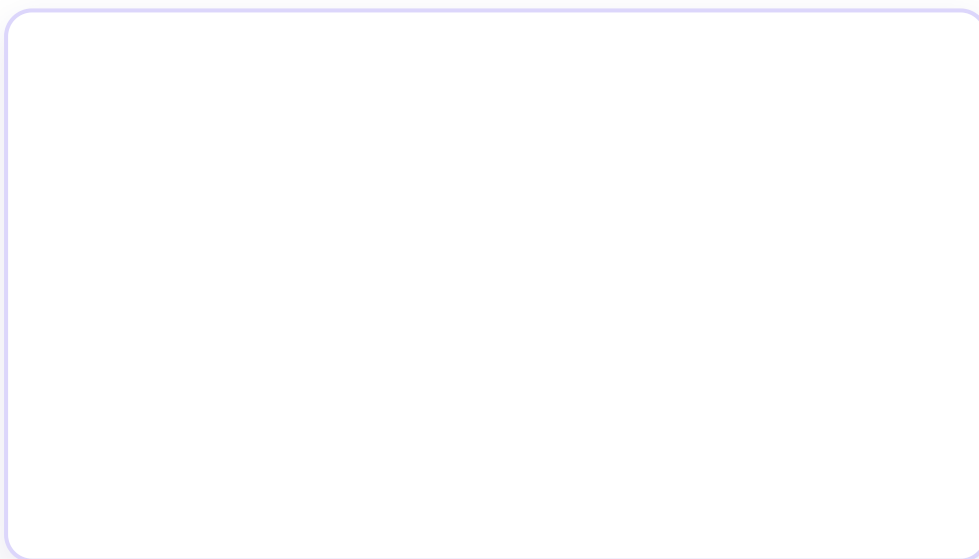
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Example 4: finding sine from cosine

Suppose $\cos \theta = \frac{4}{5}$ and θ is in quadrant IV. Then $\sin^2 \theta = 1 - \frac{16}{25} = \frac{9}{25}$, so $\sin \theta = \pm \frac{3}{5}$. Sine is negative in quadrant IV, so $\sin \theta = -\frac{3}{5}$ and $\tan \theta = \frac{-3/5}{4/5} = -\frac{3}{4}$.

5. Graphs of sine and cosine

Unwrapping the unit circle onto a number line gives the graphs below. As θ goes once around the circle, the height $\sin \theta$ rises from 0 to 1, falls to -1 and returns to 0. The pattern then repeats forever.



Trigonometric functions grade 11 : graphs of $y = \sin x$ in violet and $y = \cos x$ in orange over one period from 0 to 2π with their key points

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Properties of $y = \sin x$ and $y = \cos x$

Domain: all real numbers. Range: $[-1, 1]$. Period: 2π . The sine graph passes through the origin and is symmetric about it (an odd function); the cosine graph crosses the y-axis at 1 and is symmetric about the y-axis (an even function). Each graph is described by five key points per period, spaced a quarter period apart.

6. Amplitude, period, phase shift and vertical shift

The general sinusoidal function has the form $y = A \sin(B(x - C)) + D$, and the same pattern holds for cosine. Each letter transforms the basic graph.

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Reading the parameters

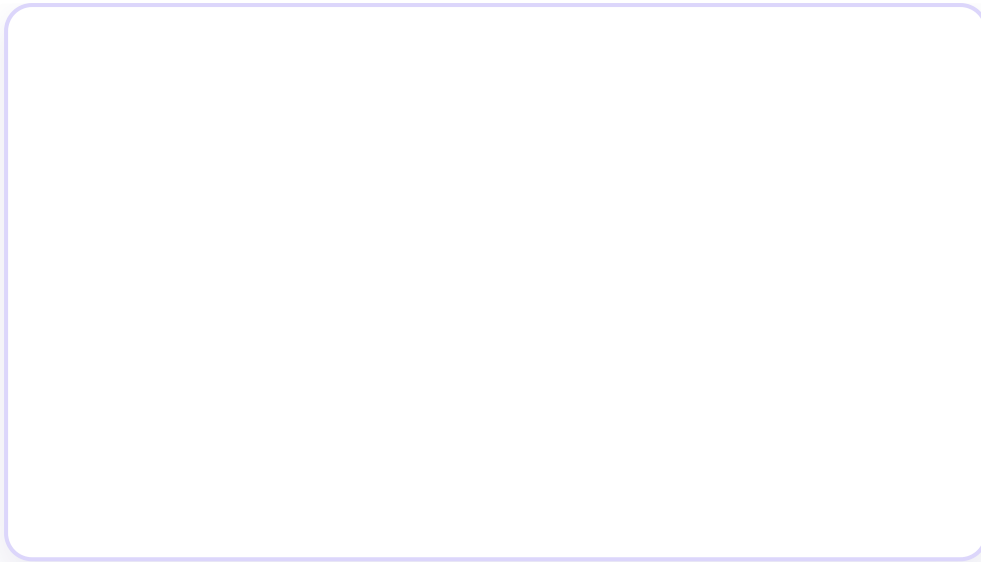
Amplitude $|A|$: half the distance between the maximum and the minimum. Period $\frac{2\pi}{|B|}$: the length of one full cycle. Phase shift C : a move of C units to the right (to the left if $C < 0$). Vertical shift D : the midline is $y = D$, so the maximum is $D + |A|$ and the minimum is $D - |A|$. A negative A reflects the graph over the midline.

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Example 5: graphing a transformed sine wave

Graph $y = 2 \sin\left(2\left(x - \frac{\pi}{4}\right)\right) + 1$.

Amplitude 2, period $\frac{2\pi}{2} = \pi$, phase shift $\frac{\pi}{4}$ to the right, midline $y = 1$. So the maximum is 3 and the minimum is -1 . A cycle begins at $x = \frac{\pi}{4}$ on the midline, then the quarter-period steps of $\frac{\pi}{4}$ give: maximum at $\frac{\pi}{2}$, midline at $\frac{3\pi}{4}$, minimum at π , midline at $\frac{5\pi}{4}$.



Trigonometric functions grade 11 : orange sine wave with amplitude 2, period π , shift π over 4 to the right and midline $y = 1$, compared with the dashed basic sine curve

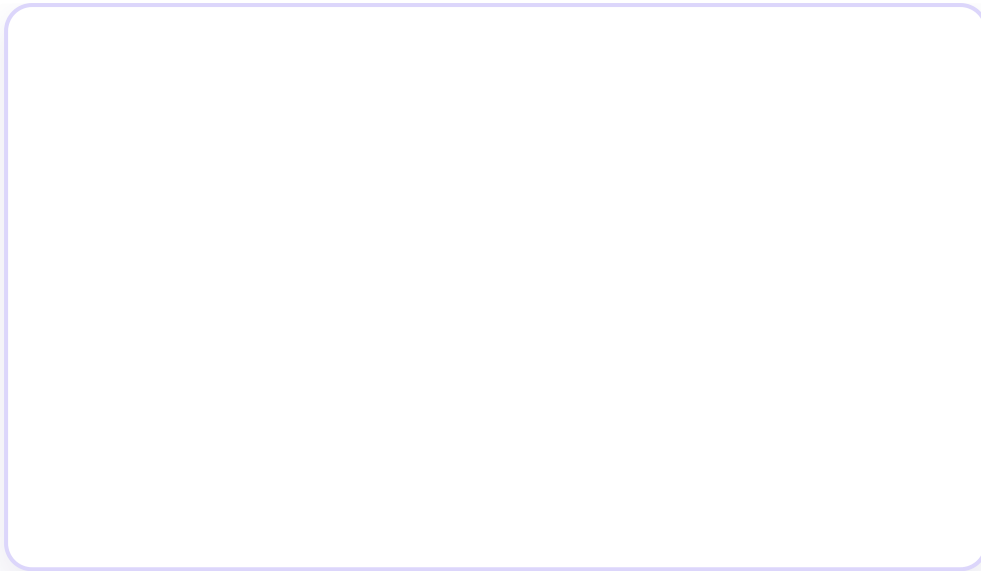


Common mistake

In $y = \sin(2x - \pi)$, the shift is not π . Factor out the 2: $\sin\left(2\left(x - \frac{\pi}{2}\right)\right)$, so the shift is $\frac{\pi}{2}$.

7. The graph of tangent and its asymptotes

Since $\tan x = \frac{\sin x}{\cos x}$, the tangent is undefined wherever $\cos x = 0$, that is, at $x = \frac{\pi}{2} + k\pi$ for any integer k . Near those values the graph shoots toward $\pm\infty$, so each one gives a vertical asymptote.



Trigonometric functions grade 11 : graph of the tangent function with vertical asymptotes at $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ and the points at $-\frac{\pi}{4}$, 0 and $\frac{\pi}{4}$

P Properties of $y = \tan x$

The period is π , the range is all real numbers (there is no amplitude), the zeros are $x = k\pi$, and the graph is increasing on each branch. For $y = \tan(Bx)$ the period is $\frac{\pi}{|B|}$ and the asymptotes are at $x = \frac{\pi}{2|B|} + \frac{k\pi}{|B|}$.

8. Solving basic trigonometric equations

M Method: solving on an interval such as $[0, 2\pi)$

1. Isolate the trigonometric expression, for example $\sin x = \frac{1}{2}$.
2. Find the reference angle from the exact values table.
3. Use the sign of the value to choose the quadrants, then list every solution in the interval.
4. For all real solutions, add the period: $2k\pi$ for sine and cosine, $k\pi$ for tangent.

E**Example 6: three equations**

(a) $\sin x = -\frac{\sqrt{2}}{2}$ on $[0, 2\pi)$: the reference angle is $\frac{\pi}{4}$ and sine is negative in quadrants III and IV, so $x = \frac{5\pi}{4}$ or $x = \frac{7\pi}{4}$.

(b) $\tan x = 1$ on $[0, 2\pi)$: $x = \frac{\pi}{4}$ or $x = \frac{5\pi}{4}$; all real solutions are $x = \frac{\pi}{4} + k\pi$.

(c) $2\sin^2 x - \sin x = 0$: factor to $\sin x (2\sin x - 1) = 0$. Then $\sin x = 0$ gives $x = 0, \pi$ and $\sin x = \frac{1}{2}$ gives $x = \frac{\pi}{6}, \frac{5\pi}{6}$.

**Common mistake**

Never divide both sides by $\sin x$ in part (c): you would lose the solutions where $\sin x = 0$. Factor instead. Also remember that an equation like $\sin x = \frac{1}{2}$ has two solutions per period, not one.

9. Modeling periodic phenomena

When data repeat smoothly, a sinusoid $y = A \cos(B(t - C)) + D$ is a good model. Read the parameters directly from the situation.

M**Method: building a model**

1. Amplitude: $A = \frac{\text{max} - \text{min}}{2}$. Midline: $D = \frac{\text{max} + \text{min}}{2}$.
2. Period P : then $B = \frac{2\pi}{P}$.
3. Choose cosine for a maximum at the start (or $-\cos$ for a minimum at the start), then use the shift C if the cycle begins later.
4. Check the model against two known data points.

E**Example 7: a bicycle pedal**

The pedal of a bike is 4 inches above the ground at its lowest point and 18 inches at its highest. The crank makes one full turn every 2 seconds, starting at the lowest point at $t = 0$. The amplitude is $\frac{18 - 4}{2} = 7$, the midline is $\frac{18 + 4}{2} = 11$, and $B = \frac{2\pi}{2} = \pi$. Starting at a minimum means a reflected cosine: $h(t) = 11 - 7\cos(\pi t)$. Check: $h(0) = 4$ and $h(1) = 18$. At $t = 0.5$ second, $h = 11$ inches, which is about 28 centimeters.

Key takeaways

- $180^\circ = \pi$ radians; convert with the factors $\frac{\pi}{180}$ and $\frac{180}{\pi}$. Arc length is $s = r\theta$ with θ in radians.
- On the unit circle, the point at angle θ is $(\cos \theta, \sin \theta)$ and $\tan \theta = \frac{\sin \theta}{\cos \theta}$.
- Use the reference angle and the quadrant sign to get exact values; learn the table for $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$.
- $\sin^2 \theta + \cos^2 \theta = 1$ for every angle.
- For $y = A \sin(B(x - C)) + D$: amplitude $|A|$, period $\frac{2\pi}{|B|}$, phase shift C , midline $y = D$.
- $\tan x$ has period π , zeros at $k\pi$, and asymptotes at $\frac{\pi}{2} + k\pi$.
- To solve an equation, find the reference angle, use the quadrants, list solutions in the interval, then add periods; factor instead of dividing.
- To model, take A from the max and min, D as their average and $B = \frac{2\pi}{\text{period}}$.



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