

Trigonometric Functions and the Unit Circle: math lesson, Grade 12 – download the PDF

Satellites orbit, Ferris wheels turn, sound waves repeat and a spinning drill bit cuts: all of these follow circles. In Grade 12 you move trigonometry off the triangle and onto the **unit circle**, where every angle, even a negative one or one bigger than a full turn, gets a sine, a cosine and a tangent. Let us build the whole picture step by step.

1. Radian measure and arc length

D Radian

One **radian** is the measure of the central angle that cuts off an arc whose length equals the radius of the circle. Since a full circle has circumference $2\pi r$, a full turn measures 2π radians, so $180^\circ = \pi$ radians.

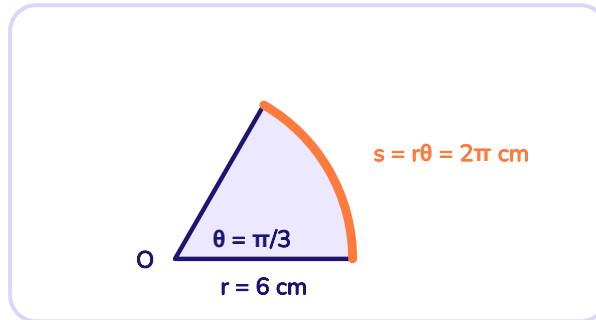
M Converting between degrees and radians

1. Degrees to radians: multiply by $\frac{\pi}{180}$.
2. Radians to degrees: multiply by $\frac{180}{\pi}$.

P

Arc length and sector area

On a circle of radius r , a central angle θ measured **in radians** intercepts an arc of length $s = r\theta$ and bounds a sector of area $A = \frac{1}{2}r^2\theta$.



E

Example 1: arc length

A sector has radius **6 cm** and central angle $\frac{\pi}{3}$. Then $s = 6 \cdot \frac{\pi}{3} = 2\pi \approx 6.28$ cm, which is about **2.47** inches.

E

Example 2: converting

$$225^\circ = 225 \cdot \frac{\pi}{180} = \frac{5\pi}{4} \text{ and } \frac{7\pi}{12} \text{ rad} = \frac{7\pi}{12} \cdot \frac{180}{\pi} = 105^\circ.$$

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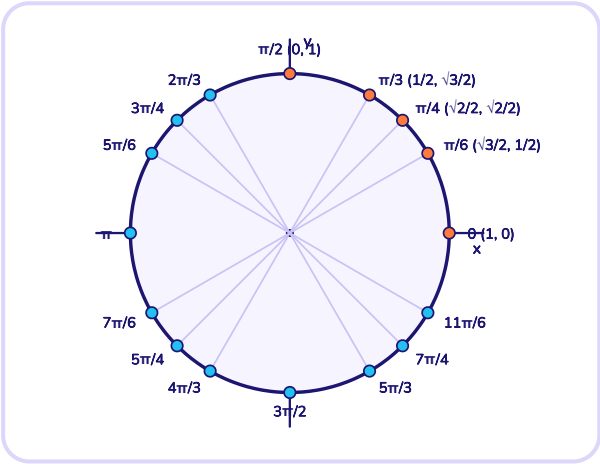
Watch out

The formula $s = r\theta$ is only true when θ is in radians. With degrees you must convert first.

2. The unit circle and its special values

D Unit circle

The **unit circle** is the circle of radius 1 centered at the origin, with equation $x^2 + y^2 = 1$. Start at the point $(1, 0)$ and travel along the circle through a signed angle θ (counterclockwise is positive). The point you reach is $(\cos \theta, \sin \theta)$.



Because the point lies on the circle, the coordinates always satisfy $\cos^2 \theta + \sin^2 \theta = 1$. Memorize the first-quadrant values; every other special angle is a reflection of them.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined



Zyro's memory trick

On my planet we write the sines as $\frac{\sqrt{0}}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$ and the cosines in the opposite order. Nothing else to remember!

The sign of each coordinate depends on the quadrant:

Quadrant	Angle range	sin	cos	tan
I	$0 < \theta < \frac{\pi}{2}$	+	+	+
II	$\frac{\pi}{2} < \theta < \pi$	+	-	-
III	$\pi < \theta < \frac{3\pi}{2}$	-	-	+
IV	$\frac{3\pi}{2} < \theta < 2\pi$	-	+	-

3. The six trigonometric functions

D Six functions

For the point $(x, y) = (\cos \theta, \sin \theta)$ on the unit circle:

$$\sin \theta = y, \quad \cos \theta = x, \quad \tan \theta = \frac{y}{x} = \frac{\sin \theta}{\cos \theta},$$

$$\csc \theta = \frac{1}{\sin \theta}, \quad \sec \theta = \frac{1}{\cos \theta}, \quad \cot \theta = \frac{\cos \theta}{\sin \theta}.$$

A function is undefined wherever its denominator is 0: **tan** and **sec** at $\theta = \frac{\pi}{2} + k\pi$, and **csc** and **cot** at $\theta = k\pi$.

P Pythagorean identities

$$\sin^2 \theta + \cos^2 \theta = 1, \quad 1 + \tan^2 \theta = \sec^2 \theta, \quad 1 + \cot^2 \theta = \csc^2 \theta.$$

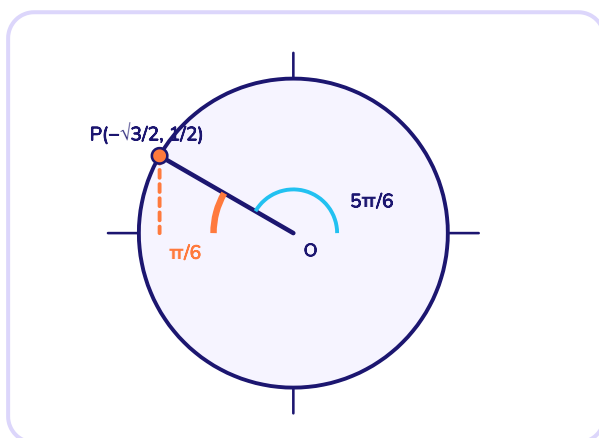
E Example 3: all six values from one

Suppose $\sin \theta = \frac{5}{13}$ and θ is in quadrant II. Then $\cos^2 \theta = 1 - \frac{25}{169} = \frac{144}{169}$, and cosine is negative in quadrant II, so $\cos \theta = -\frac{12}{13}$. Therefore $\tan \theta = -\frac{5}{12}$, $\csc \theta = \frac{13}{5}$, $\sec \theta = -\frac{13}{12}$ and $\cot \theta = -\frac{12}{5}$.

4. Reference angles

D Reference angle

The **reference angle** of θ is the acute angle between the terminal side of θ and the x -axis. In quadrant II it is $\pi - \theta$, in quadrant III it is $\theta - \pi$, and in quadrant IV it is $2\pi - \theta$.



M Evaluating with a reference angle

1. Reduce θ to an angle between 0 and 2π .
2. Find its quadrant and its reference angle.
3. Take the value of the function at the reference angle.
4. Attach the sign that the function has in that quadrant.

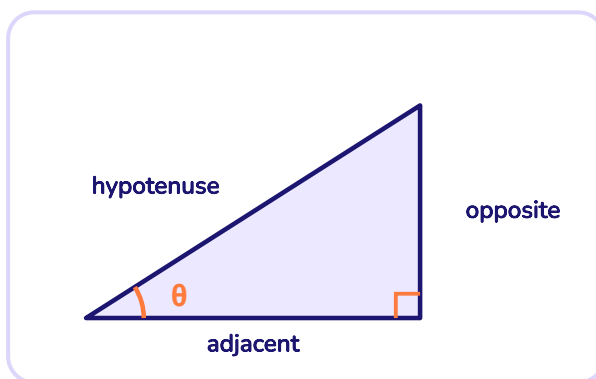
E Example 4: a third-quadrant angle

$\theta = \frac{7\pi}{6}$ is in quadrant III and its reference angle is $\frac{7\pi}{6} - \pi = \frac{\pi}{6}$. Sine and cosine are both negative there, so $\sin \frac{7\pi}{6} = -\frac{1}{2}$ and $\cos \frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$. Tangent is positive:

$$\tan \frac{7\pi}{6} = \frac{\sqrt{3}}{3}.$$

5. Right triangle trigonometry

For an acute angle θ in a right triangle, the unit circle values become ratios of sides.



P SOH-CAH-TOA

$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$, $\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$, $\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$. The two acute angles are complementary and satisfy $\sin \theta = \cos \left(\frac{\pi}{2} - \theta \right)$.

E Example 5: a kite

A kite string **80 ft** long makes a **55°** angle with level ground and is pulled straight. The kite's height is **$80 \sin 55^\circ \approx 65.5$ ft**, about **20.0 m**.

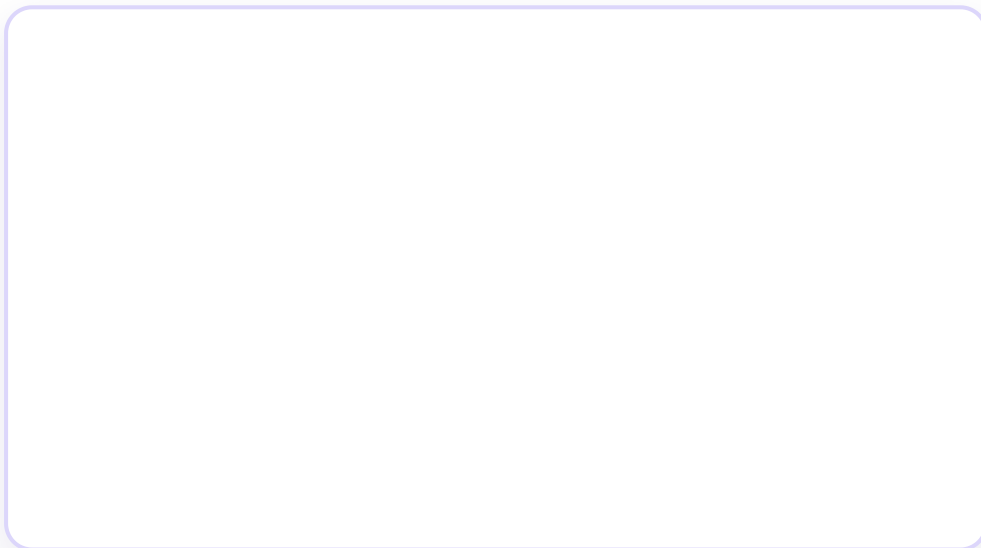


Check your mode

Set your calculator to degree mode for 55° and to radian mode for $\frac{\pi}{3}$. Most wrong answers come from the wrong mode.

6. Periodicity and symmetry

Going once around the circle brings you back to the same point, so adding 2π to an angle changes nothing.



Unit circle grade 12: graphs of sine in blue and cosine in orange over one period from 0 to 2π with ticks every π over 2



Periodicity and symmetry

- $\sin(\theta + 2\pi) = \sin \theta$ and $\cos(\theta + 2\pi) = \cos \theta$: period 2π . Also $\tan(\theta + \pi) = \tan \theta$: period π .
- Cosine is **even**: $\cos(-\theta) = \cos \theta$. Sine and tangent are **odd**: $\sin(-\theta) = -\sin \theta$, $\tan(-\theta) = -\tan \theta$.
- $\sin(\pi - \theta) = \sin \theta$, $\cos(\pi - \theta) = -\cos \theta$, $\sin(\theta + \pi) = -\sin \theta$.

E Example 6: big and negative angles

$\sin \frac{19\pi}{6}$: subtract $2\pi = \frac{12\pi}{6}$ to get $\frac{7\pi}{6}$, so the value is $-\frac{1}{2}$. And
 $\cos \left(-\frac{7\pi}{3}\right) = \cos \frac{7\pi}{3} = \cos \frac{\pi}{3} = \frac{1}{2}$.

7. Angular and linear velocity

D Angular and linear velocity

An object moving on a circle of radius r sweeps an angle θ in time t . Its **angular velocity** is $\omega = \frac{\theta}{t}$ (radians per unit of time) and its **linear velocity** is $v = \frac{s}{t} = r\omega$.

One revolution per minute (rpm) equals 2π radians per minute, that is $\frac{\pi}{30}$ radians per second.

E Example 7: a spinning disk

A disk of radius **6 cm** spins at **480 rpm**. Then $\omega = 480 \cdot \frac{2\pi}{60} = 16\pi \approx 50.27 \text{ rad/s}$, and the rim moves at $v = 6 \cdot 16\pi = 96\pi \approx 301.6 \text{ cm/s}$, or about **3.02 m/s**.



Same ω , different speeds

Every point of a turning wheel shares the same angular velocity, but points farther from the center travel faster because $v = r\omega$.

Key takeaways

- π radians = 180° ; arc length is $s = r\theta$ and sector area is $\frac{1}{2}r^2\theta$ with θ in radians.
- The point at angle θ on the unit circle is $(\cos \theta, \sin \theta)$, so $\cos^2 \theta + \sin^2 \theta = 1$.
- Know the values at $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$; get the others with reference angles and quadrant signs.
- $\tan = \frac{\sin}{\cos}$, and **csc**, **sec**, **cot** are the reciprocals of **sin**, **cos**, **tan**.
- In a right triangle: SOH-CAH-TOA.
- Sine and cosine have period 2π ; tangent has period π . Cosine is even; sine and tangent are odd.
- $\omega = \frac{\theta}{t}$ and $v = r\omega$; convert rpm with $1 \text{ rpm} = 2\pi \text{ rad/min}$.



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