

Trigonometric Identities and Equations: math lesson, Grade 12 – download the PDF

An equation is true for a few values; an **identity** is true for every value where both sides make sense. Trigonometric identities are the toolkit that lets you rewrite a messy expression into a clean one, find exact values such as **$\sin 15^\circ$** , and solve equations that look impossible at first. In this chapter you will build the whole family step by step, starting from one single fact about the unit circle.

1. Reciprocal and quotient identities

D Identity

A trigonometric **identity** is an equation that holds for every angle at which both sides are defined.

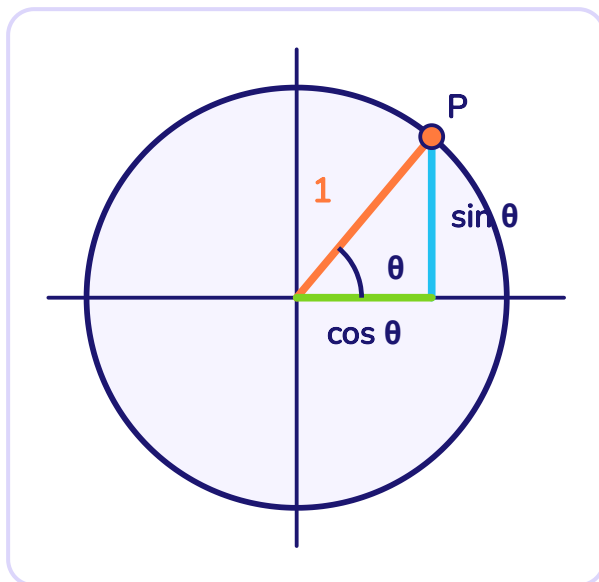
Everything starts with the six trigonometric functions. For any angle θ where the denominators are not zero:

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \quad \cot \theta = \frac{\cos \theta}{\sin \theta}, \quad \sec \theta = \frac{1}{\cos \theta}, \quad \csc \theta = \frac{1}{\sin \theta}.$$

These are the **quotient** and **reciprocal** identities. When you get stuck on a proof, rewriting every function in terms of sine and cosine is often the winning first move.

2. The Pythagorean identities

Take a point P on the unit circle at angle θ . Its coordinates are $(\cos \theta, \sin \theta)$, and the segment from the origin to P has length 1.



P Pythagorean identities

For every angle θ :

$$\sin^2 \theta + \cos^2 \theta = 1$$

and, wherever the functions are defined,

$$1 + \tan^2 \theta = \sec^2 \theta, \quad 1 + \cot^2 \theta = \csc^2 \theta.$$

The first identity is the Pythagorean theorem applied to the right triangle in the figure: legs $\cos \theta$ and $\sin \theta$, hypotenuse 1. Divide it by $\cos^2 \theta$ to get the second one, and by $\sin^2 \theta$ to get the third. It can also be written $\sin^2 \theta = 1 - \cos^2 \theta$ or $\cos^2 \theta = 1 - \sin^2 \theta$.

E**Example 1: finding the other functions**

Suppose $\sin \theta = \frac{3}{5}$ and θ is in Quadrant II. Then $\cos^2 \theta = 1 - \frac{9}{25} = \frac{16}{25}$, so $\cos \theta = \pm \frac{4}{5}$. In Quadrant II cosine is negative, so $\cos \theta = -\frac{4}{5}$ and $\tan \theta = \frac{3/5}{-4/5} = -\frac{3}{4}$.

**Watch the sign**

The square root gives two values. Always use the quadrant of the angle to choose the sign; never drop the $\pm$ silently.

3. Sum and difference formulas

P**Sum and difference formulas**

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Notice that the sign in the cosine formula is *reversed*: cosine of a sum has a minus sign. These formulas let you build exact values for angles such as 15° , 75° or 105° out of 30° , 45° and 60° .

E**Example 2: an exact value**

Write $75^\circ = 45^\circ + 30^\circ$. Then

$$\cos 75^\circ = \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

**Classic mistake**

$\sin(A + B)$ is not $\sin A + \sin B$. Test with $A = B = 30^\circ$: $\sin 60^\circ \approx 0.866$, but $\sin 30^\circ + \sin 30^\circ = 1$.

4. Double-angle formulas

Put $B = A$ in the sum formulas and you get the double-angle formulas at once.

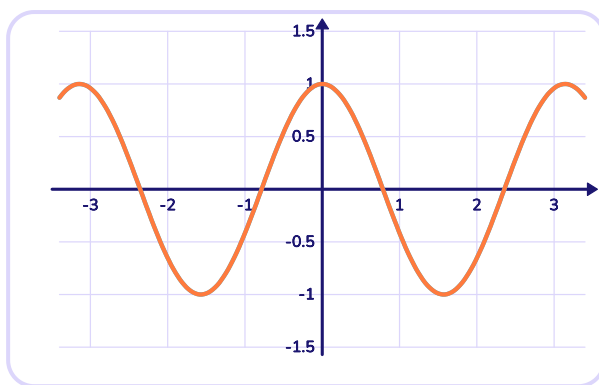
P**Double-angle formulas**

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

The three forms of $\cos 2A$ are connected by $\sin^2 A + \cos^2 A = 1$. Pick the form that contains only the function you already know. The next graph shows that $\cos 2x$ and $2 \cos^2 x - 1$ are literally the same curve.



E Example 3: doubling an angle

If $\sin A = \frac{3}{5}$ and A is in Quadrant I, then $\cos A = \frac{4}{5}$. So $\sin 2A = 2 \cdot \frac{3}{5} \cdot \frac{4}{5} = \frac{24}{25}$,
 $\cos 2A = 1 - 2 \cdot \frac{9}{25} = \frac{7}{25}$ and $\tan 2A = \frac{24}{7}$.

I 5. Half-angle formulas

Solve the forms $\cos 2A = 1 - 2 \sin^2 A$ and $\cos 2A = 2 \cos^2 A - 1$ for the squares, then replace $2A$ by θ .

P Half-angle formulas

$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}, \quad \cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}$$

$$\tan \frac{\theta}{2} = \frac{1 - \cos \theta}{\sin \theta} = \frac{\sin \theta}{1 + \cos \theta}$$

The sign is chosen by the quadrant of $\theta/2$.

E**Example 4: sine of 22.5 degrees**

Since $22.5^\circ = \frac{45^\circ}{2}$ lies in Quadrant I, the sign is positive:

$$\sin 22.5^\circ = \sqrt{\frac{1 - \frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2 - \sqrt{2}}}{2} \approx 0.383.$$

6. Product-to-sum and sum-to-product

Adding or subtracting the sum and difference formulas makes the cross terms cancel and gives formulas that turn products into sums.

P**Product-to-sum formulas**

$$\sin A \cos B = \frac{1}{2} [\sin(A + B) + \sin(A - B)]$$

$$\cos A \cos B = \frac{1}{2} [\cos(A + B) + \cos(A - B)]$$

$$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$$

Read from right to left, they become the sum-to-product formulas, for instance

$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$ and $\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$. Sum-to-product is very useful for equations, because a product equal to zero is easy to solve.

E Example 5: a product without a calculator

$$\sin 75^\circ \cos 15^\circ = \frac{1}{2}[\sin 90^\circ + \sin 60^\circ] = \frac{1}{2}\left(1 + \frac{\sqrt{3}}{2}\right) = \frac{2 + \sqrt{3}}{4} \approx 0.933.$$

7. Verifying identities

To **verify** an identity you must show that the two sides are equal for all allowed angles. You are not solving; you are transforming one side into the other.

M Method: verifying an identity

1. Work on the more complicated side only, and leave the other side untouched.
2. Rewrite everything in terms of sine and cosine, or use a Pythagorean identity.
3. Combine fractions over a common denominator, factor, or multiply by a conjugate.
4. Stop when you reach the other side. Never move terms across the equal sign.

E Example 6: verifying

Show that $\frac{\sec x - \cos x}{\tan x} = \sin x$. Left side: $\sec x - \cos x = \frac{1 - \cos^2 x}{\cos x} = \frac{\sin^2 x}{\cos x}$.
Dividing by $\tan x = \frac{\sin x}{\cos x}$ gives $\frac{\sin^2 x}{\cos x} \cdot \frac{\cos x}{\sin x} = \sin x$. The two sides match.

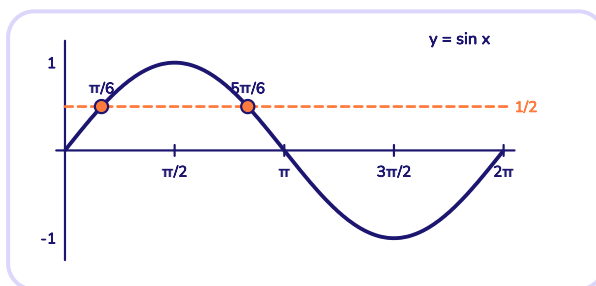


Zyro's tip

Earthlings, a false identity can be exposed with one counterexample: plug in an angle such as $\frac{\pi}{6}$ and compare. A true identity needs a proof, but a quick test tells you whether it is worth proving.

8. Solving trigonometric equations

An equation such as $\sin x = \frac{1}{2}$ is true only for certain angles. On $[0, 2\pi)$ there are two solutions, because the line $y = \frac{1}{2}$ crosses the sine curve twice.



Over all real numbers the solutions repeat every 2π : $x = \frac{\pi}{6} + 2\pi k$ or $x = \frac{5\pi}{6} + 2\pi k$, where k is any integer.



Method: solving a trigonometric equation

1. Use identities to get a single trigonometric function of a single angle.
2. Treat that function like an unknown: factor, or use the quadratic formula.
3. Solve each simple equation such as $\cos x = c$ with the unit circle.
4. List the angles in the requested interval, then check the solutions if you squared or divided.

E Example 7: a quadratic in cosine

Solve $2 \cos^2 x + \cos x - 1 = 0$ on $[0, 2\pi)$. Factor: $(2 \cos x - 1)(\cos x + 1) = 0$, so $\cos x = \frac{1}{2}$ or $\cos x = -1$. Hence $x = \frac{\pi}{3}, \frac{5\pi}{3}$ or $x = \pi$.



Never divide by a trig function

In $\sin 2x = \cos x$, dividing by $\cos x$ loses the solutions where $\cos x = 0$. Factor instead: $\cos x (2 \sin x - 1) = 0$.

Key takeaways

- $\sin^2 \theta + \cos^2 \theta = 1$, $1 + \tan^2 \theta = \sec^2 \theta$, $1 + \cot^2 \theta = \csc^2 \theta$; choose the sign from the quadrant.
- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ and $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$.
- $\sin 2A = 2 \sin A \cos A$ and $\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$.
- Half-angle: $\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}$, $\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}$.
- Product-to-sum and sum-to-product turn products into sums and back, which helps with equations.
- To verify an identity, transform one side into the other; do not solve it.
- To solve an equation, reduce to one function, factor, use the unit circle, and check your answers.



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