

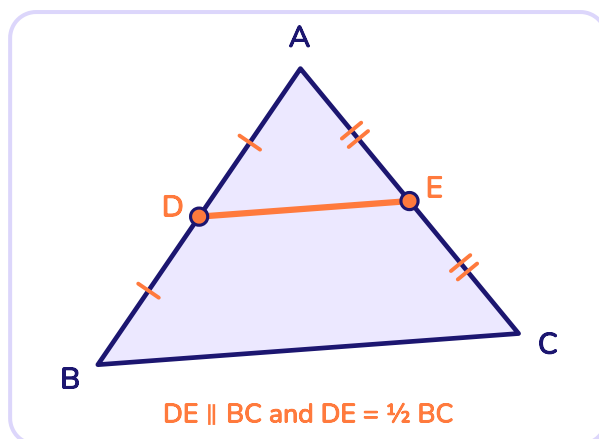
Relationships in Triangles: math lesson, Grade 10 – download the PDF

Draw any triangle and, hidden inside it, you will find special points that answer real questions. Where should a fire station go so it is the same distance from three towns? Where can you balance a triangular tray on one fingertip? Each of these points comes from a family of lines drawn in the triangle: perpendicular bisectors, angle bisectors, medians and altitudes. In this chapter you will learn what these points are, how to locate them, and what they guarantee about lengths. You will also learn two rules that link the sides and angles of a triangle: the triangle inequality, and the way the biggest angle always faces the longest side.

1. The midsegment theorem

D Midsegment of a triangle

A **midsegment** of a triangle is a segment that joins the midpoints of two sides. Every triangle has three midsegments.



P Midsegment Theorem

The midsegment of a triangle is **parallel** to the third side and is **half as long** as that side.

In triangle ABC , if D is the midpoint of $\overline{AB}$ and E is the midpoint of $\overline{AC}$, then

$$\overline{DE} \parallel \overline{BC} \quad \text{and} \quad DE = \frac{1}{2} BC.$$

E Example 1

In triangle ABC , D and E are the midpoints of $\overline{AB}$ and $\overline{AC}$. We know $DE = 3x + 1$ and $BC = 8x - 6$. Find x , DE and BC .

By the Midsegment Theorem, $BC = 2 DE$, so $8x - 6 = 2(3x + 1) = 6x + 2$. Then $2x = 8$ and $x = 4$.

Therefore $DE = 3(4) + 1 = 13$ and $BC = 8(4) - 6 = 26$. Check: $26 = 2 \times 13$.



Common mistake

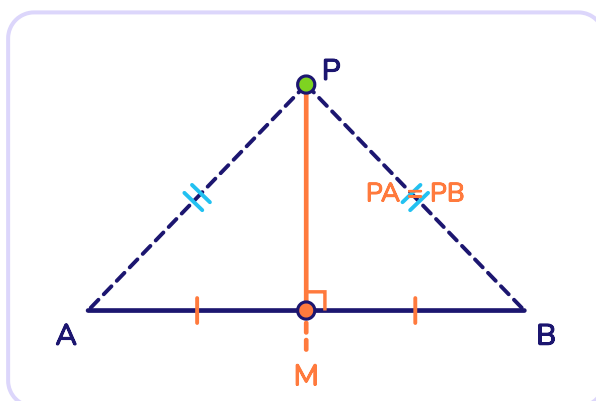
The midsegment is half of the side it is **parallel to**, not half of the sides it touches. In the figure, DE is half of BC , not half of AB .

2. Perpendicular bisectors and angle bisectors



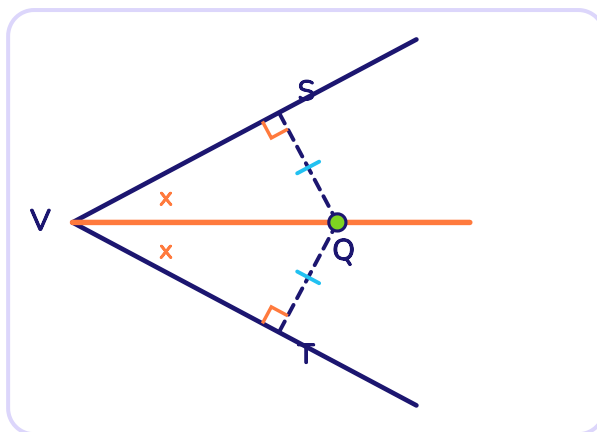
Two kinds of bisectors

The **perpendicular bisector** of a segment is the line that is perpendicular to the segment at its midpoint. An **angle bisector** is the ray that splits an angle into two equal angles.



Perpendicular Bisector Theorem and its converse

A point is on the perpendicular bisector of a segment **if and only if** it is the same distance from the two endpoints of the segment. In the figure, P is on the bisector of $\overline{AB}$, so $PA = PB$.



P Angle Bisector Theorem and its converse

A point is on the bisector of an angle **if and only if** it is the same distance from the two sides of the angle. Distance to a side always means the length of the **perpendicular** segment, so in the figure $QS = QT$.

E Example 2

Point P lies on the perpendicular bisector of $\overline{AB}$, with $PA = 5x - 4$ and $PB = 3x + 8$. Find PA .

Since P is on the perpendicular bisector, $PA = PB$: $5x - 4 = 3x + 8$, so $2x = 12$ and $x = 6$.

Then $PA = 5(6) - 4 = 26$. Check: $PB = 3(6) + 8 = 26$.

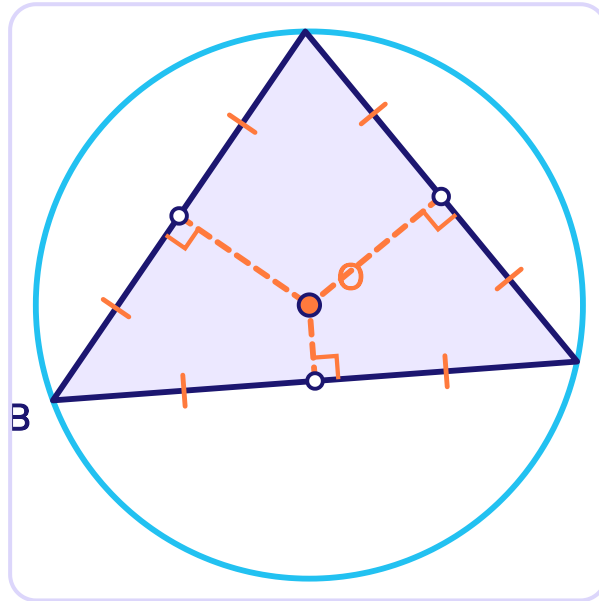
3. The circumcenter and the incenter

A triangle has three sides, so it has three perpendicular bisectors and three angle bisectors. The surprise is that each family meets at a **single point**. Lines that pass through one common point are called **concurrent**.

P

Circumcenter Theorem

The three perpendicular bisectors of a triangle are concurrent at the **circumcenter**. This point is equally far from the three vertices, so it is the center of the **circumscribed circle**, the circle through all three vertices.

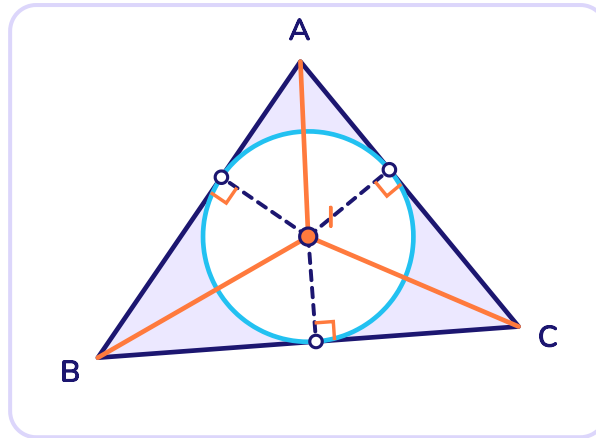


The circumcenter does not always stay inside the triangle. It is inside an acute triangle, it is exactly the **midpoint of the hypotenuse** of a right triangle, and it is outside an obtuse triangle.

P

Incenter Theorem

The three angle bisectors of a triangle are concurrent at the **incenter**. This point is equally far from the three sides, so it is the center of the **inscribed circle**, the circle that touches all three sides. The incenter is **always inside** the triangle.



E Example 3

A right triangle has legs **6** in and **8** in. Find the radius of its circumscribed circle and of its inscribed circle.

The hypotenuse is $\sqrt{6^2 + 8^2} = 10$ in. The circumcenter is the midpoint of the hypotenuse, so the circumradius is $10 \div 2 = 5$ in.

The area is $\frac{1}{2}(6)(8) = 24$ in² and the semiperimeter is $\frac{6 + 8 + 10}{2} = 12$ in. The inradius is $\text{area} \div \text{semiperimeter} = 24 \div 12 = 2$ in.

4. Medians and the centroid

D Median

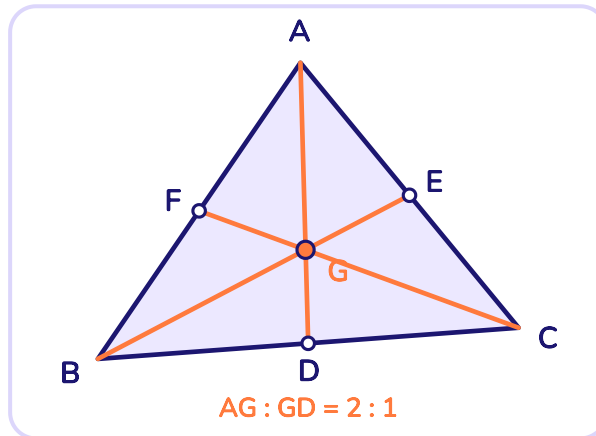
A **median** of a triangle is a segment from a vertex to the midpoint of the opposite side.

P

Centroid Theorem

The three medians of a triangle are concurrent at the **centroid**, which is always inside the triangle. The centroid is located **two thirds** of the way along each median, measured from the vertex. If G is the centroid and $\overline{AD}$ is a median, then

$$AG = \frac{2}{3} AD \quad \text{and} \quad GD = \frac{1}{3} AD, \quad \text{so } AG = 2GD.$$



The centroid is the **balance point** of a triangular plate of uniform thickness. On a coordinate plane its coordinates are just the averages of the vertex coordinates:

$$G \left(\frac{x_A + x_B + x_C}{3}, \frac{y_A + y_B + y_C}{3} \right).$$

E**Example 4**

Find the centroid of the triangle with vertices $A(-2, 1)$, $B(6, 3)$ and $C(2, 8)$. If a median has length **15** cm, how far is the centroid from its vertex?

$$x = \frac{-2 + 6 + 2}{3} = 2 \text{ and } y = \frac{1 + 3 + 8}{3} = 4, \text{ so } G(2, 4).$$

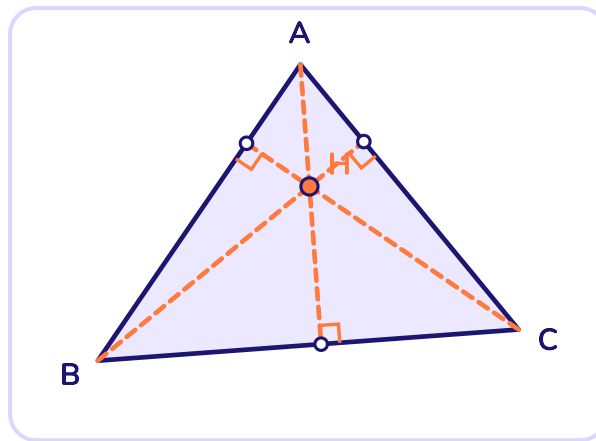
For a median of length **15** cm, the centroid is at $\frac{2}{3}(15) = 10$ cm from the vertex and 5 cm from the midpoint of the side.

5. Altitudes and the orthocenter**D****Altitude**

An **altitude** of a triangle is the perpendicular segment from a vertex to the line that contains the opposite side. Its length is the **height** of the triangle for that side.

P**Orthocenter Theorem**

The three lines that contain the altitudes of a triangle are concurrent at the **orthocenter**. It is inside an acute triangle, at the vertex of the right angle in a right triangle, and outside an obtuse triangle.



M Method: orthocenter on the coordinate plane

1. Compute the slope of one side.
2. The altitude to that side has the **opposite reciprocal** slope and passes through the opposite vertex.
3. Write the equation of two altitudes.
4. Solve the system: the solution is the orthocenter. Check with the third altitude.

E Example 5

Find the orthocenter of the triangle with $A(0, 0)$, $B(6, 0)$ and $C(2, 4)$.

$\overline{AB}$ is horizontal, so the altitude from C is the vertical line $x = 2$.

The slope of $\overline{BC}$ is $\frac{4 - 0}{2 - 6} = -1$, so the altitude from A has slope **1**: $y = x$. At $x = 2$, $y = 2$, so $H(2, 2)$.

Check with the altitude from B : the slope of $\overline{AC}$ is **2**, so that altitude has slope $-\frac{1}{2}$ and equation $y = -\frac{1}{2}(x - 6)$. At $x = 2$ it gives $y = 2$. It passes through H .

6. The four centers at a glance

Center	Made from	Always inside?	What it gives you
Circumcenter	Perpendicular bisectors of the sides	No (acute: inside, right: on the hypotenuse, obtuse: outside)	Same distance from the 3 vertices
Incenter	Angle bisectors	Yes	Same distance from the 3 sides
Centroid	Medians	Yes	Balance point, splits medians 2 : 1
Orthocenter	Lines containing the altitudes	No (same pattern as the circumcenter)	Meeting point of the heights

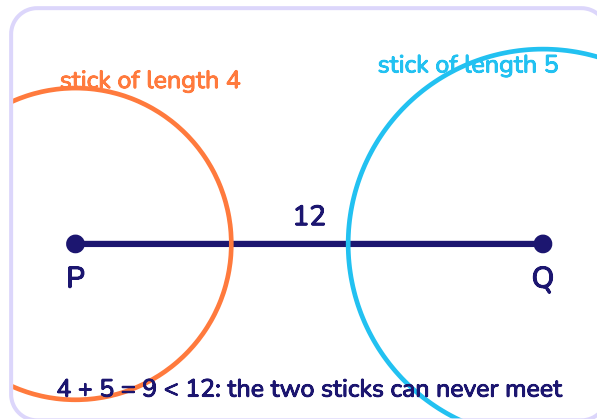


Zyro's memory trick

“Circum” means “around”: the circumcenter is the center of the circle drawn **around** the triangle. “In” means “inside”: the incenter is the center of the circle that fits **inside** it.

7. The triangle inequality theorem

Can you build a triangle from any three lengths? Try it with sticks of lengths **4**, **5** and **12**. The two short sticks, laid end to end, only reach **9**, which is less than **12**, so they can never meet to close the triangle.



P Triangle Inequality Theorem

The sum of the lengths of any two sides of a triangle is **greater than** the length of the third side. Three lengths form a triangle exactly when this holds for all three pairs.

M Method

1. Sort the lengths from smallest to largest.
2. Add the two smallest.
3. If their sum is greater than the largest, a triangle exists. If it is equal or smaller, it does not.

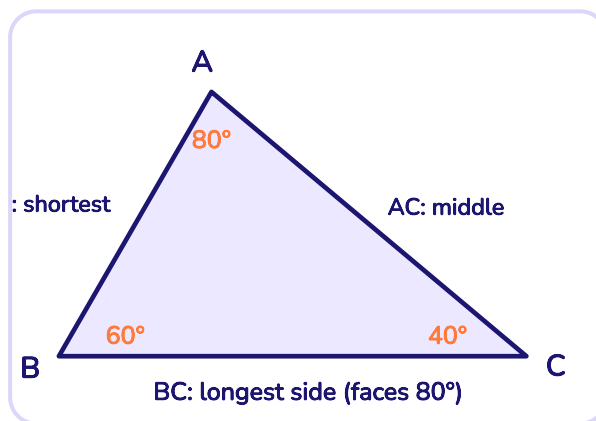
If two sides measure a and b , the third side x must satisfy $|a - b| < x < a + b$.

E**Example 6**

Two sides of a triangle measure **9** cm and **14** cm. What are the possible lengths of the third side?

The third side x must be less than $9 + 14 = 23$ and greater than $14 - 9 = 5$. So $5 < x < 23$.

A length of **5** cm would give a flat triangle with no area, so the bounds are strict.

8. Inequalities within one triangle**P****Side-Angle Relationships**

In a triangle, the **longest side is opposite the largest angle** and the **shortest side is opposite the smallest angle**. Conversely, the largest angle is opposite the longest side. In general, if one side is longer than another, the angle opposite it is larger.

E**Example 7**

In triangle XYZ , $\angle X = 52^\circ$, $\angle Y = 67^\circ$ and $\angle Z = 61^\circ$. List the sides from shortest to longest.

Each side faces the angle at the opposite vertex: $\overline{YZ}$ faces $\angle X = 52^\circ$, $\overline{XY}$ faces $\angle Z = 61^\circ$ and $\overline{XZ}$ faces $\angle Y = 67^\circ$.

Since $52 < 61 < 67$, the order is $YZ < XY < XZ$.

**Common mistake**

Always match a side with the angle **opposite** to it, not the angle at one of its ends.
The side $\overline{BC}$ goes with $\angle A$, never with $\angle B$.

Key takeaways

- The **midsegment** joins two midpoints: it is parallel to the third side and half as long.
- A point on a **perpendicular bisector** is equidistant from the segment's endpoints; a point on an **angle bisector** is equidistant from the angle's sides.
- The **circumcenter** (perpendicular bisectors) is the center of the circle through the vertices; the **incenter** (angle bisectors) is the center of the circle inside the triangle.
- The **centroid** (medians) is always inside and sits $\frac{2}{3}$ of the way from each vertex:
 $AG = 2GD$.
- The **orthocenter** (altitudes) is inside, on, or outside the triangle depending on whether it is acute, right or obtuse.
- Triangle inequality: $|a - b| < x < a + b$ for the third side.
- The longest side faces the largest angle; the shortest side faces the smallest angle.



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