

Sequences, Series, and Induction: math lesson, Grade 12 – download the PDF

A savings plan that adds the same amount every month, a rumor that doubles in reach every day, a ball that loses height on every bounce: all of them are described by lists of numbers that follow a rule. In this chapter you will learn to write that rule, add up long lists quickly, decide when an infinite list has a finite total, and prove statements that hold for every whole number. You will finish with a powerful shortcut for expanding powers of a binomial.

1. Sequences and notation

D Definition: sequence

A **sequence** is an ordered list of numbers $a_1, a_2, a_3, \dots$. The number a_n is the **n th term**, and n is its **index**. A sequence can be defined **explicitly**, by a formula for a_n in terms of n , or **recursively**, by giving the first term(s) and a rule that builds each term from the previous ones.

For instance, $a_n = n^2 + 1$ is explicit: $a_4 = 17$ immediately. By contrast, $a_1 = 3$ and $a_n = a_{n-1} + 5$ is recursive: you must find $a_2 = 8$, then $a_3 = 13$, and so on. A **series** is the sum of the terms of a sequence. The sum of the first n terms is called the **partial sum** and is written $S_n = a_1 + a_2 + \dots + a_n$.

2. Arithmetic sequences and series

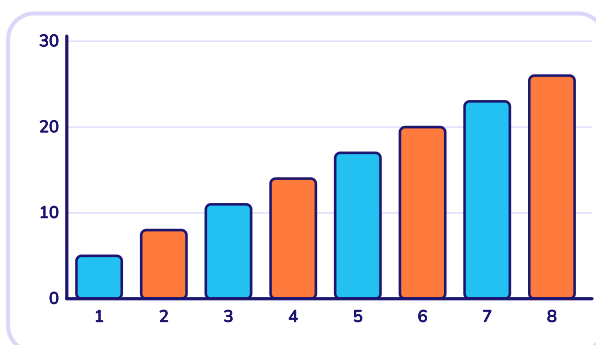
D Definition: arithmetic sequence

A sequence is **arithmetic** if consecutive terms differ by a constant d , called the **common difference**: $a_{n+1} - a_n = d$ for every n .

P Formulas for an arithmetic sequence

$$a_n = a_1 + (n - 1)d \quad \text{and} \quad S_n = \frac{n(a_1 + a_n)}{2} = \frac{n(2a_1 + (n - 1)d)}{2}.$$

The sum formula comes from writing the sum forwards and backwards. Each of the n vertical pairs adds up to $a_1 + a_n$, so twice the sum is $n(a_1 + a_n)$. The picture below shows the sequence 5, 8, 11, 14, ... : every bar is 3 units taller than the previous one.



E Example 1

An arithmetic sequence has $a_1 = 7$ and $d = 4$. Find a_{20} and S_{20} .

$$a_{20} = 7 + 19 \times 4 = 83. \text{ Then } S_{20} = \frac{20(7 + 83)}{2} = 10 \times 90 = 900.$$

3. Geometric sequences and series

D Definition: geometric sequence

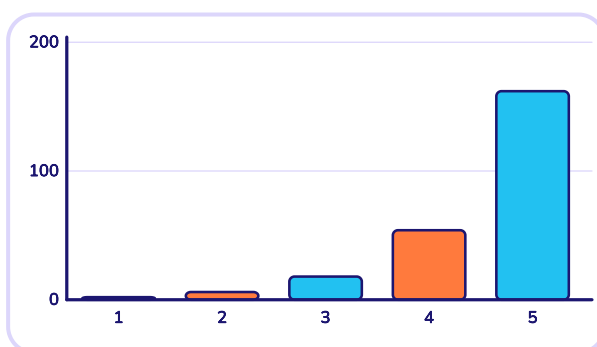
A sequence is **geometric** if each term is the previous one multiplied by a constant $r \neq 0$, the **common ratio**: $a_{n+1} = r a_n$.

P Formulas for a geometric sequence

$$a_n = a_1 r^{n-1} \quad \text{and, for } r \neq 1, \quad S_n = a_1 \frac{1 - r^n}{1 - r}.$$

If $r = 1$, every term equals a_1 and $S_n = n a_1$.

To see where the sum formula comes from, compute $S_n - rS_n$: almost everything cancels and you are left with $a_1 - a_1 r^n$, so $S_n(1 - r) = a_1(1 - r^n)$. The bars below show 2, 6, 18, 54, 162 (ratio 3): geometric growth quickly outruns arithmetic growth.



E

Example 2

Find the sum of the first 6 terms of the geometric sequence 5, 15, 45, ...

Here $a_1 = 5$ and $r = 3$. So $S_6 = 5 \cdot \frac{3^6 - 1}{3 - 1} = 5 \cdot \frac{728}{2} = 1820$. Check by adding:
 $5 + 15 + 45 + 135 + 405 + 1215 = 1820$.

4. Infinite geometric series

What happens to $S_n = a_1 \frac{1 - r^n}{1 - r}$ when n gets huge? If $|r| < 1$, then r^n shrinks toward 0, so S_n settles down to a single number. If $|r| \geq 1$, the terms do not shrink and the partial sums do not settle.

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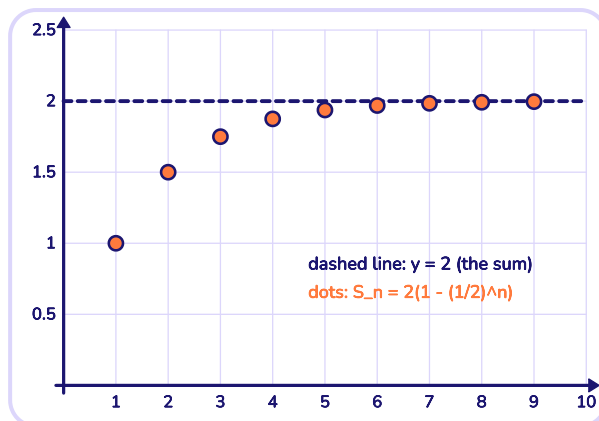
Sum of an infinite geometric series

If $|r| < 1$, the series $a_1 + a_1r + a_1r^2 + \cdots$ **converges** and its sum is

$$S = \frac{a_1}{1 - r}.$$

If $|r| \geq 1$ (and $a_1 \neq 0$), the series **diverges**: it has no finite sum.

The graph shows the partial sums of $1 + \frac{1}{2} + \frac{1}{4} + \cdots$. The dots climb toward the dashed line $y = 2$, which is exactly $\frac{1}{1 - 1/2}$.



E Example 3

Evaluate $12 - 4 + \frac{4}{3} - \dots$

The first term is 12 and the ratio is $r = -\frac{4}{12} = -\frac{1}{3}$. Since $|r| < 1$, the series converges: $S = \frac{12}{1 - (-1/3)} = \frac{12}{4/3} = 9$.



Common mistake

Never use $\frac{a_1}{1 - r}$ without checking $|r| < 1$ first. For $3 + 6 + 12 + \dots$ the formula would give -3 , which is nonsense for a sum of positive numbers.

5. Sigma notation



Definition: sigma notation

The symbol $\sum_{k=m}^n a_k$ means $a_m + a_{m+1} + \dots + a_n$. The letter k is the **index of summation**, m is the lower limit and n the upper limit.

P**Rules for sums**

$$\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k, \quad \sum_{k=1}^n c a_k = c \sum_{k=1}^n a_k, \quad \sum_{k=1}^n c = nc.$$

Useful formulas: $\sum_{k=1}^n k = \frac{n(n+1)}{2}$ and $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$.

E**Example 4**

Evaluate $\sum_{k=1}^5 (2k + 1)$ two ways.

Expanding: $3 + 5 + 7 + 9 + 11 = 35$. With the rules: $2 \cdot \frac{5 \cdot 6}{2} + 5 = 30 + 5 = 35$.

6. Recursive definitions

A recursive definition needs two ingredients: **initial term(s)** and a **recurrence relation**. An arithmetic sequence is $a_n = a_{n-1} + d$, a geometric one is $a_n = r a_{n-1}$. Many sequences are neither, for example the Fibonacci-style rule $f_n = f_{n-1} + f_{n-2}$, which needs *two* starting values.

M**Method: from a recurrence to a formula**

1. Compute the first four or five terms with the recurrence.
2. Look for a pattern (constant differences, constant ratios, powers, squares).
3. Write a conjecture for a_n .
4. Prove it by induction (next section).

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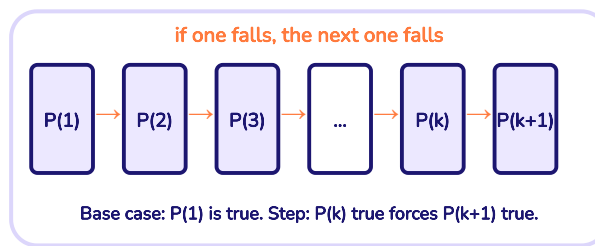
Example 5

Let $a_1 = 3$ and $a_n = 2a_{n-1} - 1$. The terms are 3, 5, 9, 17. Each is one more than a power of 2, so we conjecture $a_n = 2^n + 1$. Check the recurrence:

$2(2^{n-1} + 1) - 1 = 2^n + 1$. The formula is consistent.

7. Mathematical induction

Suppose you want to prove that a statement $P(n)$ is true for every integer $n \geq 1$. You cannot test infinitely many cases, but you can use the domino idea shown below: if the first domino falls, and each falling domino knocks over the next, then they all fall.



P

Principle of mathematical induction

Let $P(n)$ be a statement about an integer n . If

1. **Base case:** $P(1)$ is true, and
2. **Inductive step:** for every $k \geq 1$, if $P(k)$ is true (the **inductive hypothesis**), then $P(k+1)$ is true,

then $P(n)$ is true for all integers $n \geq 1$.

E

Example 6

Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for all $n \geq 1$.

Base case. For $n = 1$: the left side is 1 and the right side is $1^2 = 1$.

Inductive step. Assume $1 + 3 + \cdots + (2k - 1) = k^2$. Then

$1 + 3 + \cdots + (2k - 1) + (2k + 1) = k^2 + 2k + 1 = (k + 1)^2$, which is the statement for $n = k + 1$.

By induction, the formula holds for every $n \geq 1$.

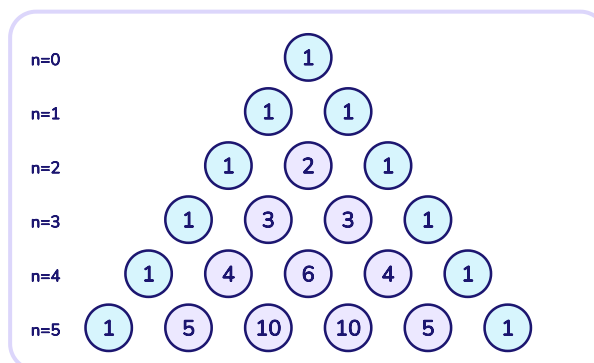
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Zyro's tip

On my home planet we say: the inductive step must *use* the hypothesis. If your proof of $P(k + 1)$ never mentions $P(k)$, something is missing!

8. The Binomial Theorem

Expanding $(a + b)^n$ by hand is slow, but the coefficients follow a pattern. They are the **binomial coefficients** $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, the number of ways to choose k objects from n . They form **Pascal's triangle**: each entry is the sum of the two entries above it.



Binomial Theorem

For every integer $n \geq 0$,

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k.$$

The expansion has $n + 1$ terms, and the coefficients add up to 2^n .

Example 7

Expand $(x + 2)^4$.

Row 4 of Pascal's triangle is 1, 4, 6, 4, 1. So

$$(x + 2)^4 = x^4 + 4(2)x^3 + 6(4)x^2 + 4(8)x + 16 = x^4 + 8x^3 + 24x^2 + 32x + 16.$$

Common mistake

Do not forget to raise the second term to the power too, including its coefficient and its sign: in $(x - 2)^4$ the powers of -2 alternate in sign.

Key takeaways

- Arithmetic: $a_n = a_1 + (n - 1)d$ and $S_n = \frac{n(a_1 + a_n)}{2}$.
- Geometric: $a_n = a_1 r^{n-1}$ and $S_n = a_1 \frac{1 - r^n}{1 - r}$ for $r \neq 1$.
- An infinite geometric series converges only if $|r| < 1$, and then $S = \frac{a_1}{1 - r}$.
- Sigma notation packs a sum into one line, and sums split over addition and constant multiples.
- A recursive definition needs a starting value and a rule; conjecture a formula, then prove it.
- Induction: prove the base case, then prove that $P(k)$ implies $P(k + 1)$.
- Binomial Theorem: $(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$, with coefficients from Pascal's triangle.



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