

Sequences and Series: math lesson, Grade 11 – download the PDF

Patterns that grow, shrink, or repeat are everywhere: a savings plan that adds the same amount each month, a ball that loses height with every bounce, a loan balance that changes by a fixed percent. In this chapter you will learn to describe such patterns with **sequences**, to add up their terms with **series**, and to write long sums compactly with **sigma notation**.

1. What is a sequence?

D Sequence

A **sequence** is an ordered list of numbers called **terms**. We write $a_1, a_2, a_3, \dots$ where a_1 is the first term and a_n is the term in position n (the *general term*). A sequence is a function whose inputs are the positive integers $1, 2, 3, \dots$

There are two common ways to define a sequence.

- An **explicit formula** gives a_n directly from n . For example, $a_n = n^2 - 1$ gives $0, 3, 8, 15, \dots$, and you can jump straight to a_{50} .
- A **recursive formula** gives the first term and a rule that builds each term from the previous one, such as $a_1 = 4$ and $a_{n+1} = a_n + 3$, which gives $4, 7, 10, 13, \dots$

2. Arithmetic sequences

D Arithmetic sequence

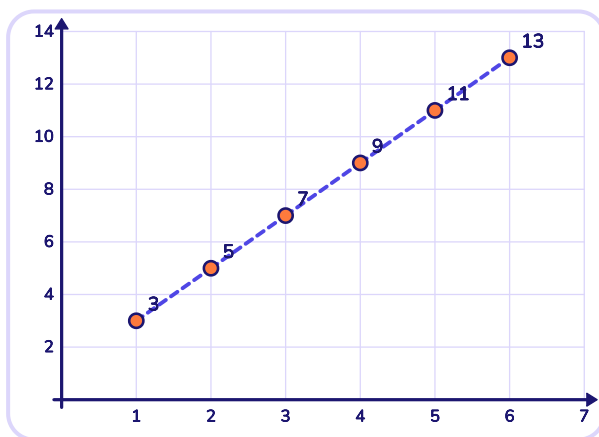
A sequence is **arithmetic** if the difference between consecutive terms is always the same number d , the **common difference**: $a_{n+1} - a_n = d$.

P Explicit formula

$$a_n = a_1 + (n - 1)d$$

Recursive form: $a_{n+1} = a_n + d$.

The points of an arithmetic sequence lie on a straight line, and the common difference is its slope.



E Example 1: finding a term

Consider $7, 12, 17, 22, \dots$. The common difference is $d = 12 - 7 = 5$, so
 $a_n = 7 + (n - 1) \cdot 5 = 5n + 2$. The 20th term is $a_{20} = 5 \cdot 20 + 2 = 102$.

3. Geometric sequences

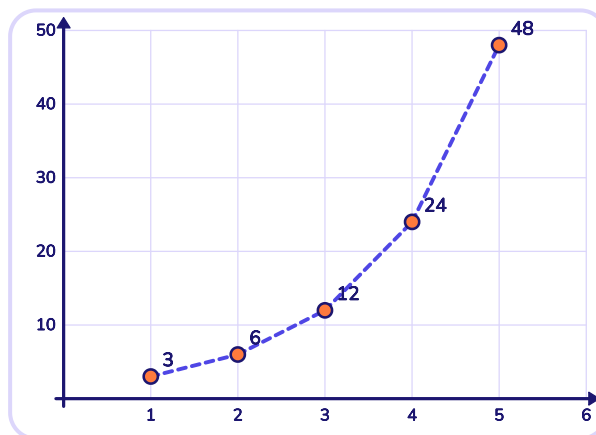
D Geometric sequence

A sequence is **geometric** if each term is the previous one multiplied by the same nonzero number r , the **common ratio**: $\frac{a_{n+1}}{a_n} = r$.

P Explicit formula

$$a_n = a_1 \cdot r^{n-1}$$

Recursive form: $a_{n+1} = r \cdot a_n$.



E Example 2: a geometric sequence

For $3, 6, 12, 24, \dots$ the ratio is $r = 6 \div 3 = 2$, so $a_n = 3 \cdot 2^{n-1}$. Then $a_8 = 3 \cdot 2^7 = 3 \cdot 128 = 384$.



Careful

The exponent is $n - 1$, not n . Check with $n = 1$: you must get a_1 back. Also, a ratio between 0 and 1 makes the terms shrink, and a negative ratio makes them alternate in sign.

4. Explicit and recursive formulas



Method: switching between the two forms

1. Decide whether the pattern is arithmetic (constant difference) or geometric (constant ratio).
2. Find a_1 and d (or r).
3. Recursive form: write a_1 and $a_{n+1} = a_n + d$ (or $a_{n+1} = r a_n$).
4. Explicit form: use $a_n = a_1 + (n - 1)d$ (or $a_n = a_1 r^{n-1}$).
5. Test both forms on the first three terms.

If you know two terms of an arithmetic sequence, subtract them and divide by the number of steps between them to get d . For a geometric sequence, divide them and take the root of the same order to get r .

5. Arithmetic series

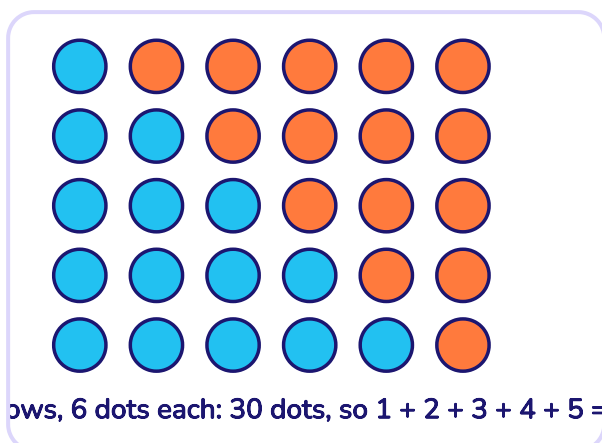


Series

A **series** is the sum of the terms of a sequence. The **partial sum**

$S_n = a_1 + a_2 + \cdots + a_n$ is the sum of the first n terms.

To add $1 + 2 + 3 + 4 + 5$, put a copy of the same dots upside down next to the first one. You get a rectangle of 5 rows and 6 columns, so the double of the sum is $5 \cdot 6$.



P Sum of an arithmetic series

$$S_n = \frac{n(a_1 + a_n)}{2} \quad \text{or} \quad S_n = \frac{n[2a_1 + (n-1)d]}{2}$$

In words: the number of terms times the average of the first and last terms.

E Example 3: a sum of 15 terms

Add $5 + 9 + 13 + \dots$ up to 15 terms. Here $d = 4$ and $a_{15} = 5 + 14 \cdot 4 = 61$. Then

$$S_{15} = \frac{15(5 + 61)}{2} = \frac{15 \cdot 66}{2} = 495.$$

6. Finite geometric series

P Sum of a finite geometric series

For $r \neq 1$:

$$S_n = a_1 \frac{1 - r^n}{1 - r}$$

If $r = 1$, every term equals a_1 and $S_n = n a_1$.

Why it works. Multiply $S_n = a_1 + a_1 r + \dots + a_1 r^{n-1}$ by r and subtract:

$S_n - rS_n = a_1 - a_1 r^n$. Almost every term cancels, and dividing by $1 - r$ gives the formula.

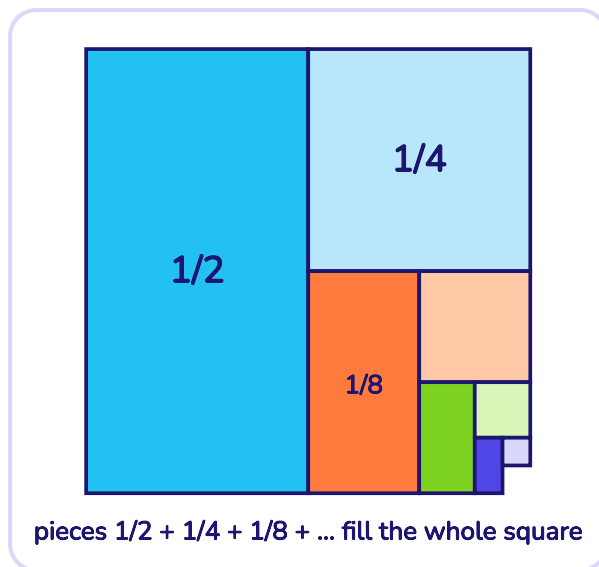
E Example 4: doubling money

A game awards 2 points for the first level, and each level pays 3 times as many points as the one before. Total for 6 levels: $a_1 = 2$, $r = 3$, so

$$S_6 = 2 \cdot \frac{1 - 3^6}{1 - 3} = 2 \cdot \frac{-728}{-2} = 728 \text{ points.}$$

7. Infinite geometric series

Can you add infinitely many numbers and get a finite answer? Yes, when the terms shrink fast enough. Cut a square of area 1 in half, then cut half of what remains, and so on: the pieces $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ fill the square, so their sum is 1.



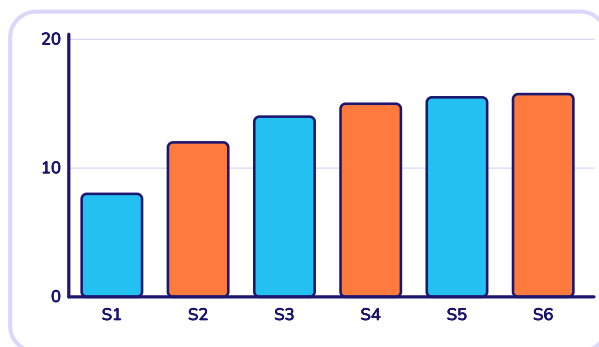
P Sum of an infinite geometric series

If $|r| < 1$, the series converges and

$$S = \frac{a_1}{1 - r}$$

If $|r| \geq 1$, the series **diverges**: it has no finite sum.

The reason is that r^n gets closer and closer to 0 when $|r| < 1$, so the partial sums $a_1 \frac{1 - r^n}{1 - r}$ approach $\frac{a_1}{1 - r}$. The chart shows the partial sums of $8 + 4 + 2 + 1 + \dots$ closing in on $\frac{8}{1 - 1/2} = 16$.



E Example 5: shrinking terms

For $12 + 4 + \frac{4}{3} + \dots$ we have $r = \frac{4}{12} = \frac{1}{3}$, and $|r| < 1$. So

$$S = \frac{12}{1 - \frac{1}{3}} = \frac{12}{\frac{2}{3}} = 18.$$

! Zyro's tip

On my home planet we test convergence first: look at the ratio. If it is not strictly between -1 and 1 , stop, because there is no sum to compute!

8. Sigma notation

D Sigma notation

$$\sum_{k=m}^n a_k = a_m + a_{m+1} + \dots + a_n$$

k is the **index**, m is the **lower limit**, and n is the **upper limit**. The number of terms is $n - m + 1$.

E**Example 6: expanding and evaluating**

$\sum_{k=1}^5 (3k - 1) = 2 + 5 + 8 + 11 + 14 = 40$. It is arithmetic with $a_1 = 2$, $a_5 = 14$, so the formula gives $\frac{5(2 + 14)}{2} = 40$. The same sum written for a geometric series is $\sum_{k=1}^{\infty} 5\left(\frac{1}{2}\right)^{k-1} = \frac{5}{1 - \frac{1}{2}} = 10$.

**Careful**

When the lower limit is not 1, count the terms with $n - m + 1$. For instance, $\sum_{k=3}^9$ has 7 terms, not 6 or 9.

Useful rules: $\sum (a_k + b_k) = \sum a_k + \sum b_k$ and $\sum c a_k = c \sum a_k$.

Key takeaways

- Arithmetic: constant difference d , $a_n = a_1 + (n - 1)d$.
- Geometric: constant ratio r , $a_n = a_1 r^{n-1}$.
- A recursive formula needs a starting term and a rule; an explicit formula gives a_n directly.
- Arithmetic series: $S_n = \frac{n(a_1 + a_n)}{2}$.
- Finite geometric series: $S_n = a_1 \frac{1 - r^n}{1 - r}$ for $r \neq 1$.
- Infinite geometric series: $S = \frac{a_1}{1 - r}$ only when $|r| < 1$.
- In sigma notation, the number of terms is upper limit minus lower limit plus 1.



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