

Sequences and Series: math lesson, College – download the PDF

Can you add up infinitely many numbers and still get a finite answer? Sometimes yes, and sometimes no. This chapter gives you the tools to decide which case you are in, and then to turn that knowledge into powerful formulas for functions such as e^x and $\ln(1+x)$.

1. Sequences and their limits

D Definition: convergent sequence

A sequence (a_n) is a list of numbers indexed by $n = 1, 2, 3, \dots$. It **converges** to L if a_n gets as close to L as you like, and stays that close, once n is large enough. We write $\lim_{n \rightarrow \infty} a_n = L$. If no such L exists, the sequence **diverges**.

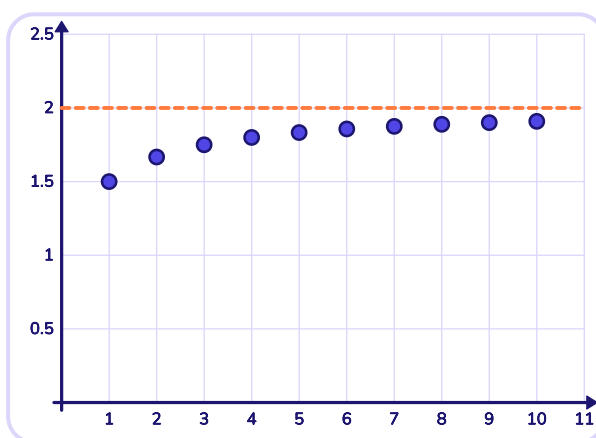
Most limits you meet are found by dividing by the highest power of n and using $\frac{1}{n} \rightarrow 0$. Two more facts help: if $a_n = f(n)$ and $f(x) \rightarrow L$ as $x \rightarrow \infty$, then $a_n \rightarrow L$; and a sequence that is bounded and monotone (always increasing or always decreasing) must converge.

E Example 1: a rational sequence

Find $\lim_{n \rightarrow \infty} \frac{4n^2 + 1}{n^2 - 3n}$. Divide the top and bottom by n^2 :

$$\frac{4 + \frac{1}{n^2}}{1 - \frac{3}{n}} \rightarrow \frac{4 + 0}{1 - 0} = 4.$$

The sequence converges to 4.



In the figure above, the terms of $a_n = \frac{2n+1}{n+1}$ climb toward the dashed line $y = 2$. Every term stays below 2, yet the gap shrinks to zero.

2. Series and geometric series

D Definition: series and partial sums

A **series** is the infinite sum $\sum_{n=1}^{\infty} a_n$. Its **partial sums** are $s_N = a_1 + a_2 + \cdots + a_N$. The series **converges** to S when $s_N \rightarrow S$; otherwise it diverges. So a series is just a sequence of partial sums.

P

Geometric series

For $a \neq 0$, the series $\sum_{n=0}^{\infty} ar^n$ converges if and only if $|r| < 1$, and then

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

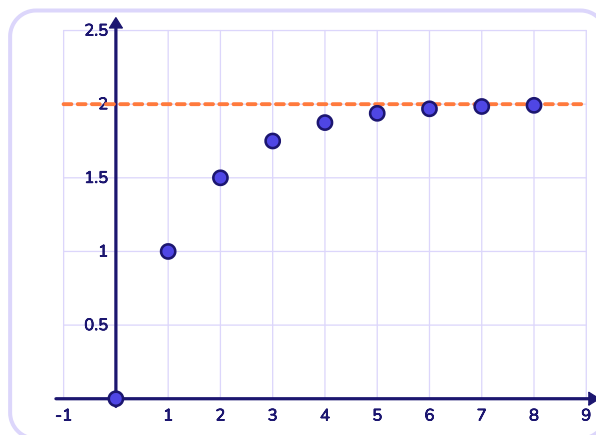
If $|r| \geq 1$ it diverges.

The reason is the identity $s_N = a \frac{1 - r^{N+1}}{1 - r}$: when $|r| < 1$ the power r^{N+1} vanishes in the limit.

E

Example 2: a geometric series

Compute $\sum_{n=0}^{\infty} 7(0.8)^n$. Here $a = 7$ and $r = 0.8$, so $|r| < 1$ and the sum is $\frac{7}{1-0.8} = 35$.



The partial sums of $1 + \frac{1}{2} + \frac{1}{4} + \cdots$ shown above creep up to 2 but never pass it.

3. The divergence test and the integral test

P Divergence test

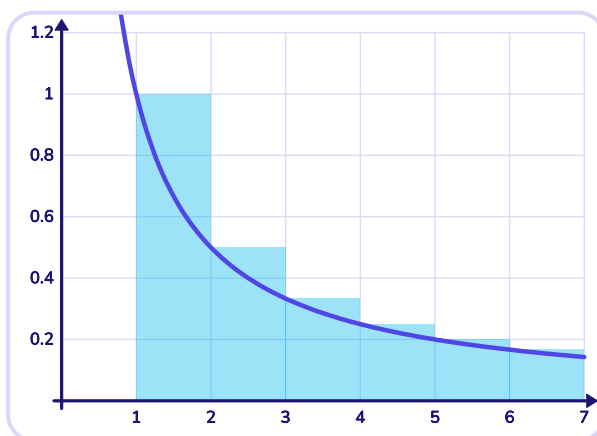
If $\sum a_n$ converges, then $a_n \rightarrow 0$. Equivalently: if a_n does not tend to 0, then $\sum a_n$ diverges.

⚠ Watch out

The test only works in one direction. If $a_n \rightarrow 0$ you learn nothing. The harmonic series $\sum \frac{1}{n}$ has terms tending to 0, yet it diverges.

P Integral test and p-series

Let f be positive, continuous and decreasing on $[1, \infty)$ with $a_n = f(n)$. Then $\sum a_n$ and $\int_1^\infty f(x) dx$ either both converge or both diverge. As a consequence, the **p-series** $\sum \frac{1}{n^p}$ converges exactly when $p > 1$.



Each rectangle in the figure has area $\frac{1}{n}$ and lies above the curve $y = \frac{1}{x}$. Their total area is at least $\int_1^7 \frac{dx}{x} = \ln 7$, and $\ln N$ grows without bound, so the harmonic series diverges (slowly: you need

over 12,000 terms to pass 10).

4. Comparison tests

P Direct and limit comparison

Let $a_n, b_n > 0$. (1) If $a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges. If $a_n \geq b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges. (2) If $\lim \frac{a_n}{b_n} = c$ with $0 < c < \infty$, then both series behave the same way.

M Method

1. Keep only the dominant terms of a_n for large n .
2. Choose a series b_n you already understand, usually a p-series or geometric series.
3. Check the inequality or compute the limit of a_n/b_n .

E Example 3: comparison

Does $\sum \frac{1}{n^2 + n + 3}$ converge? For $n \geq 1$, $n^2 + n + 3 > n^2$, so $\frac{1}{n^2 + n + 3} < \frac{1}{n^2}$. Since $\sum \frac{1}{n^2}$ is a convergent p-series ($p = 2$), the given series converges.

5. The ratio and root tests

P Ratio test and root test

Let $L = \lim \left| \frac{a_{n+1}}{a_n} \right|$ (ratio test) or $L = \lim \sqrt[n]{|a_n|}$ (root test). If $L < 1$ the series converges absolutely; if $L > 1$ (or $L = \infty$) it diverges; if $L = 1$ the test is inconclusive.

These tests shine with factorials, exponentials and n -th powers, because the messy parts cancel.

E Example 4: ratio test

For $\sum \frac{5^n}{n!}$:

$$\frac{a_{n+1}}{a_n} = \frac{5^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n} = \frac{5}{n+1} \rightarrow 0 < 1.$$

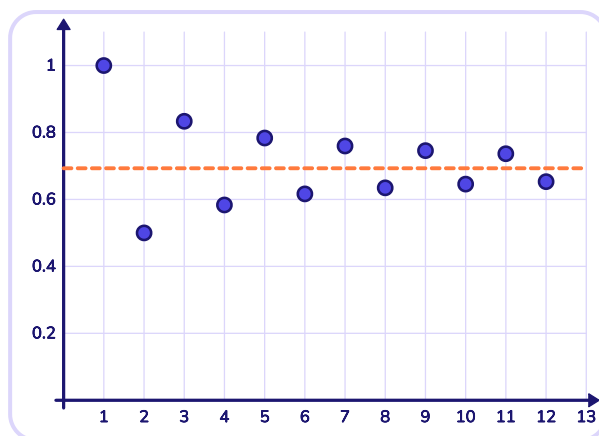
The series converges.

6. Alternating series

P Alternating series test

If $b_n > 0$, b_n is decreasing and $b_n \rightarrow 0$, then $\sum (-1)^{n+1} b_n$ converges. Moreover the error after N terms satisfies $|S - s_N| \leq b_{N+1}$.

A series converges **absolutely** if $\sum |a_n|$ converges. If it converges but not absolutely, it converges **conditionally**. Absolute convergence always implies convergence.



The partial sums of $1 - \frac{1}{2} + \frac{1}{3} - \dots$ bounce around $\ln 2 \approx 0.693$, each swing smaller than the last. This series is conditionally convergent, because $\sum \frac{1}{n}$ diverges.

7. Power series

D Definition: power series

A **power series** centered at c is $\sum_{n=0}^{\infty} c_n(x - c)^n$. It converges on an interval around c of radius R (possibly 0 or ∞): inside $(c - R, c + R)$ it converges absolutely, outside it diverges, and the endpoints must be tested one by one.

Use the ratio test on the terms to find R . Inside the interval you may differentiate and integrate term by term, which produces new power series from old ones. For instance, from $\frac{1}{1-x} = \sum x^n$ for $|x| < 1$ you get $\frac{1}{(1-x)^2} = \sum nx^{n-1}$.

8. Taylor series

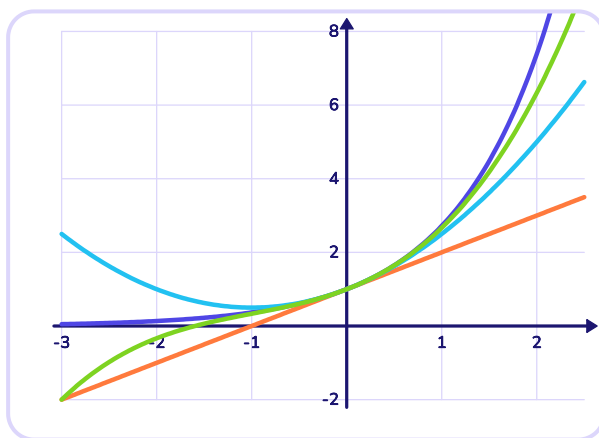
D Taylor and Maclaurin series

If f has derivatives of all orders at c , its **Taylor series** is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n.$$

When $c = 0$ it is called a **Maclaurin series**.

Three Maclaurin series are worth memorizing: $e^x = \sum \frac{x^n}{n!}$ for all x ; $\sin x = \sum (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x ; and $\ln(1+x) = \sum_{n \geq 1} (-1)^{n+1} \frac{x^n}{n}$ for $-1 < x \leq 1$.



E Example 5: approximating $e^{0.5}$

Use the first four terms of e^x at $x = 0.5$:

$$1 + 0.5 + \frac{0.5^2}{2} + \frac{0.5^3}{6} = 1 + 0.5 + 0.125 + 0.02083 \approx 1.6458.$$

A calculator gives $e^{0.5} \approx 1.6487$, so the error is about 0.003.

! Zyro's tip

On my home planet we say: to survive a series, ask three questions in order. Do the terms go to zero? Does it look like a p-series or a geometric series? Is there a factorial or an n -th power? The answer tells you the test to use.

Key takeaways

- A series converges when its sequence of partial sums has a finite limit.
- Geometric series: $\sum ar^n = \frac{a}{1-r}$ when $|r| < 1$, divergent otherwise.
- If $a_n \not\rightarrow 0$, the series diverges; if $a_n \rightarrow 0$, you still need another test.
- p-series converge exactly when $p > 1$; the integral and comparison tests extend this to similar series.
- Ratio and root tests: $L < 1$ converges, $L > 1$ diverges, $L = 1$ tells you nothing.
- Alternating series with decreasing terms tending to 0 converge, and the error is at most the first omitted term.
- Power series have a radius of convergence; test the endpoints separately.
- The Taylor series of f at c has coefficients $\frac{f^{(n)}(c)}{n!}$.



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