

Rational Expressions and Functions: math lesson, Grade 11 – download the PDF

Fractions are not only for numbers: you can build fractions out of polynomials too. In this chapter you will learn to simplify, combine and graph these “algebraic fractions,” to solve equations that contain them, and to model real situations in which one quantity goes up as another goes down. Everything rests on one habit: **always watch the denominator.**

1. Rational expressions and their domain

D Definition

A **rational expression** is a quotient of two polynomials, $\frac{P(x)}{Q(x)}$, where $Q(x)$ is not the zero polynomial. A **rational function** is a function defined by such an expression,

$$f(x) = \frac{P(x)}{Q(x)}.$$

Division by zero is impossible, so every value of x that makes the denominator **0** must be excluded. The **domain** of f is the set of all real numbers except those excluded values. To find them, set the denominator equal to zero and solve.

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Example 1: finding the domain Find the domain of $f(x) = \frac{x+6}{x^2-3x-10}$.

Factor the denominator: $x^2 - 3x - 10 = (x - 5)(x + 2)$. It equals **0** when $x = 5$ or $x = -2$. The domain is all real numbers except **-2** and **5**.

2. Simplifying rational expressions

A fraction is simplified by dividing the top and the bottom by a **common factor**. With polynomials, that means you must **factor first**, then cancel the factors that appear in both the numerator and the denominator.

M Method: simplify a rational expression

1. Factor the numerator and the denominator completely (GCF, difference of squares, trinomials).
2. List the values of x that make an original denominator 0.
3. Cancel the common factors.
4. Write what is left, keeping the restrictions from step 2.

E Example 2: simplifying Simplify $\frac{x^2 - 9}{x^2 + x - 12}$.

Factor: $\frac{(x - 3)(x + 3)}{(x + 4)(x - 3)}$. Restrictions: $x \neq -4$ and $x \neq 3$. Cancel $(x - 3)$:

$$\frac{x^2 - 9}{x^2 + x - 12} = \frac{x + 3}{x + 4}, \quad x \neq -4, x \neq 3.$$

Common mistake You may only cancel *factors* (things that are multiplied), never *terms* (things that are added). In $\frac{x + 5}{x + 7}$ you cannot “cancel the x ”: the result is not $\frac{5}{7}$. Try $x = 1$: the left side is $\frac{6}{8}$, not $\frac{5}{7}$.

3. Multiplying and dividing

Multiply rational expressions exactly like numerical fractions: top times top, bottom times bottom. To divide, multiply by the **reciprocal** of the divisor. The smart order is: factor everything, cancel, and only then multiply what remains.

P**Rules**

$$\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}, \quad \frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C} \quad (B, C, D \neq 0).$$

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Example 3: multiplying Compute $\frac{x^2 - 4}{x + 5} \cdot \frac{x^2 + 5x}{x - 2}$.

Factor: $\frac{(x - 2)(x + 2)}{x + 5} \cdot \frac{x(x + 5)}{x - 2}$. Cancel $(x - 2)$ and $(x + 5)$. What remains is $x(x + 2)$, with $x \neq -5$ and $x \neq 2$.

When you divide, remember that the divisor itself may not be zero, so its numerator also produces a restriction.

4. Adding and subtracting

You can only add fractions that share a denominator. If the denominators differ, rewrite each fraction with the **least common denominator (LCD)**: the product of every different factor, each taken with its highest power.

M**Method: add or subtract**

1. Factor each denominator and build the LCD.
2. Multiply the top and bottom of each fraction by the missing factors.
3. Add or subtract the numerators (use parentheses when subtracting!).
4. Simplify if possible.

E Example 4: adding Compute $\frac{3}{x+2} + \frac{5}{x-1}$.

The LCD is $(x+2)(x-1)$:

$$\frac{3(x-1) + 5(x+2)}{(x+2)(x-1)} = \frac{3x-3+5x+10}{(x+2)(x-1)} = \frac{8x+7}{(x+2)(x-1)}.$$

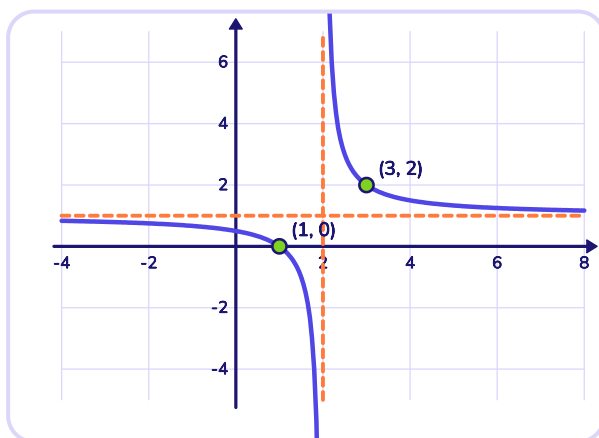
5. Rational functions and their graphs

The simplest rational function is $f(x) = \frac{1}{x}$. Its graph has two branches, one in each of two opposite quadrants, and never touches either axis. Shifting it gives the family

$$f(x) = \frac{a}{x-h} + k.$$

Here $x = h$ is the vertical line the graph cannot cross, $y = k$ is the horizontal line it approaches, and a stretches (and, if negative, flips) the branches.

E Example 5: a shifted hyperbola The function $f(x) = \frac{x-1}{x-2}$ can be rewritten as $f(x) = \frac{1}{x-2} + 1$ because $\frac{x-2+1}{x-2} = 1 + \frac{1}{x-2}$. So $h = 2$, $k = 1$, $a = 1$. The graph is the parent graph moved 2 units right and 1 unit up, as shown below; it crosses the x -axis at $(1, 0)$ and passes through $(3, 2)$.



6. Asymptotes and holes

P How to read a rational function

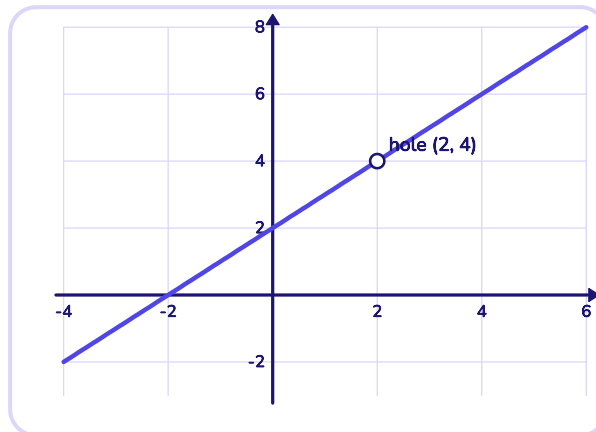
Write $f(x) = \frac{P(x)}{Q(x)}$ in lowest terms.

- A **vertical asymptote** occurs at every real zero of Q that is not canceled.
- A **hole** occurs at a value $x = a$ where a common factor $(x - a)$ was canceled. Its height is found by plugging a into the simplified expression.
- The **horizontal asymptote** depends on the degrees of P and Q , as in the table.

Degrees	Horizontal asymptote	Example
degree of top < degree of bottom	$y = 0$	$\frac{5}{x^2 + 1} \rightarrow 0$
degree of top = degree of bottom	$y = \frac{\text{leading coefficient of top}}{\text{leading coefficient of bottom}}$	$\frac{6x^2 - 1}{2x^2 + 7} : y = 3$
degree of top > degree of bottom	none (the graph grows without bound)	$\frac{x^3}{x + 1}$

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Example 6: asymptotes and a hole Look at $f(x) = \frac{x^2 - 4}{x - 2}$. Factoring gives $\frac{(x - 2)(x + 2)}{x - 2} = x + 2$ for $x \neq 2$. So the graph is the line $y = x + 2$ with a hole at $x = 2$, where the height is $2 + 2 = 4$: the hole is the point $(2, 4)$. There is no vertical asymptote.

**!**

Zyro's tip On my home planet we say: "Cancel a factor, get a hole; keep a factor, get a wall." A canceled factor leaves a hole in the graph, and a factor that survives in the denominator builds a vertical asymptote.

7. Solving rational equations

To solve an equation with fractions, multiply both sides by the LCD of every denominator. This clears the fractions and leaves a polynomial equation. Then **check every answer in the original equation**: a solution that makes a denominator zero is an **extraneous solution** and must be thrown out.

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Example 7: one solution Solve $\frac{4}{x + 1} + \frac{2}{x - 1} = \frac{10}{x^2 - 1}$.

Multiply by $(x + 1)(x - 1)$: $4(x - 1) + 2(x + 1) = 10$, so $6x - 2 = 10$ and $x = 2$.

Check: $\frac{4}{3} + \frac{2}{1} = \frac{10}{3}$ and $\frac{10}{3}$ is the right side. The solution is $x = 2$.

E **Example 8: an extraneous solution** Solve $\frac{x}{x-3} = \frac{3}{x-3} + 2$.

Multiply by $x - 3$: $x = 3 + 2(x - 3) = 2x - 3$, so $x = 3$. But $x = 3$ makes the denominator 0. The solution is extraneous, so the equation has **no solution**.

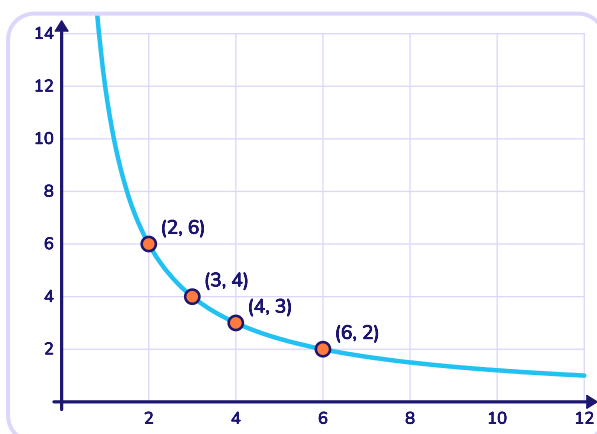
8. Variation

Variation describes how two or more quantities change together. The number k is called the **constant of variation**.

D Types of variation

- **Direct:** $y = kx$. When x doubles, y doubles.
- **Inverse:** $y = \frac{k}{x}$, that is $xy = k$. When x doubles, y is cut in half.
- **Joint:** $y = kxz$. The quantity varies directly with each of two others.
- **Combined:** a mix, such as $y = \frac{kx}{z}$.

E **Example 9: inverse variation** The time t (in hours) to empty a tank varies inversely with the number n of pumps. With $n = 4$ pumps it takes $t = 3$ hours. Then $k = nt = 12$, so $t = \frac{12}{n}$. With $n = 6$ pumps: $t = 2$ hours. The graph of $y = \frac{12}{x}$ is below.



Key takeaways

- A rational expression is a quotient of polynomials; values that make the denominator 0 are excluded from the domain.
- Factor first, then cancel *factors*, never terms.
- To multiply, factor and cancel; to divide, multiply by the reciprocal; to add or subtract, use the LCD.
- Vertical asymptote: a surviving zero of the denominator. Hole: a canceled factor. Horizontal asymptote: compare the degrees.
- Solve a rational equation by multiplying by the LCD, then reject extraneous solutions.
- Direct variation $y = kx$, inverse variation $y = \frac{k}{x}$, joint variation $y = kxz$.



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