

Radicals and Rational Exponents: math lesson, Grade 11 – download the PDF

Square roots show up when you work backward from an area, and cube roots when you work backward from a volume. In Algebra 2 you take the idea much further: roots of any order, exponents that are fractions, equations with radicals, and the curves they draw. By the end of this chapter you will move smoothly between radical form and exponent form, simplify messy expressions, and solve equations without falling for the classic trap of the **extraneous solution**.

1. nth roots

D nth root

Let n be an integer with $n \geq 2$. A number b is an **nth root** of a if $b^n = a$. The **principal nth root** of a is written $\sqrt[n]{a}$. The integer n is the **index** and a is the **radicand**.

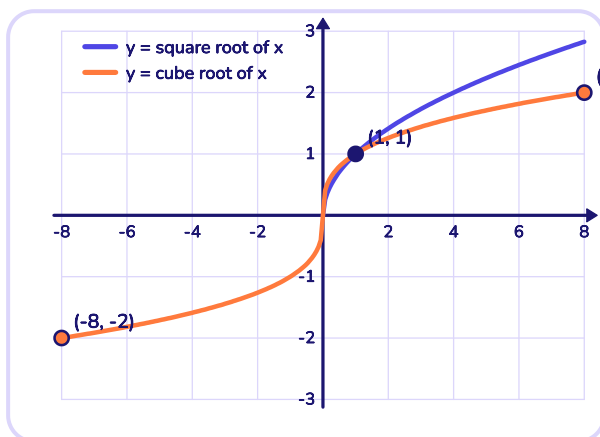
The parity of the index changes everything:

- If n is **odd**, every real number a has exactly one real n th root. For example $\sqrt[3]{-125} = -5$ because $(-5)^3 = -125$.
- If n is **even**, only $a \geq 0$ has real n th roots. The principal root is the **non-negative** one: $\sqrt{81} = 9$, not -9 , and $\sqrt[4]{16} = 2$.



Careful with even indices

$\sqrt{x^2}$ equals $|x|$, not simply x . For $x = -7$ we get $\sqrt{49} = 7 = |-7|$. With an odd index there is no problem: $\sqrt[n]{x^n} = x$ for every real x .



The figure shows the two simplest radical functions. The square root curve exists only for $x \geq 0$ and rises slowly. The cube root curve passes through the origin and extends on both sides, because negative numbers have real cube roots.

2. Rational exponents



Rational exponent

For $a > 0$ (or $a \geq 0$ when the exponent is positive), an integer $n \geq 2$ and an integer m :

$$a^{1/n} = \sqrt[n]{a} \quad \text{and} \quad a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$$

The denominator of the exponent is the **index of the root**, and the numerator is the **power**. Taking the root first keeps the numbers small, so it is usually the smart order.

E Example 1

Evaluate $16^{3/4}$ and $27^{-2/3}$.

$$16^{3/4} = \left(\sqrt[4]{16}\right)^3 = 2^3 = 8.$$

$$27^{-2/3} = \frac{1}{27^{2/3}} = \frac{1}{\left(\sqrt[3]{27}\right)^2} = \frac{1}{3^2} = \frac{1}{9}.$$

a	$a^{1/2}$	$a^{1/3}$	$a^{1/4}$	$a^{2/3}$	$a^{3/2}$
4	2	—	—	—	8
8	—	2	—	4	—
16	4	—	2	—	64
81	9	—	3	—	729
64	8	4	—	16	512

A dash means the result is not a whole number. Learn these values by heart: they make every later calculation faster.

3. Properties of exponents

Rational exponents obey exactly the same rules as integer exponents. For $a, b > 0$ and rational numbers r and s :

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Exponent rules

- $a^r \cdot a^s = a^{r+s}$ and $\frac{a^r}{a^s} = a^{r-s}$
- $(a^r)^s = a^{rs}$
- $(ab)^r = a^r b^r$ and $\left(\frac{a}{b}\right)^r = \frac{a^r}{b^r}$
- $a^{-r} = \frac{1}{a^r}$

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Example 2

Simplify $\frac{x^{5/3} \cdot x^{-1/3}}{x^{2/3}}$ for $x > 0$, and $(8x^3)^{2/3}$.

Add exponents on top: $\frac{5}{3} - \frac{1}{3} = \frac{4}{3}$. Then subtract: $\frac{4}{3} - \frac{2}{3} = \frac{2}{3}$. The result is $x^{2/3}$.

$$(8x^3)^{2/3} = 8^{2/3} \cdot (x^3)^{2/3} = 4 \cdot x^2 = 4x^2.$$

4. Simplifying radical expressions

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Product and quotient rules

For $a, b \geq 0$ (and $b \neq 0$ in a quotient):

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

A radical is **simplified** when no factor of the radicand is a perfect n th power, there is no fraction inside the radical, and no radical remains in a denominator.

M**Method**

1. Factor the radicand so that one factor is the largest perfect n th power.
2. Split the radical with the product rule.
3. Take the root of the perfect power and leave the rest inside.

E**Example 3**

Simplify $\sqrt{72}$ and $\sqrt[3]{54x^4}$.

$$\sqrt{72} = \sqrt{36 \cdot 2} = 6\sqrt{2}.$$

$$\sqrt[3]{54x^4} = \sqrt[3]{27 \cdot 2 \cdot x^3 \cdot x} = 3x\sqrt[3]{2x}.$$

To remove a radical from a denominator, multiply the top and bottom by a suitable factor. This is called **rationalizing**. For a binomial denominator use the **conjugate**, which turns the product into a difference of squares.

E**Example 4**

Rationalize $\frac{6}{\sqrt{3}}$ and $\frac{1}{2 + \sqrt{3}}$.

$$\frac{6}{\sqrt{3}} = \frac{6\sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}.$$

$$\frac{1}{2 + \sqrt{3}} = \frac{2 - \sqrt{3}}{(2 + \sqrt{3})(2 - \sqrt{3})} = \frac{2 - \sqrt{3}}{4 - 3} = 2 - \sqrt{3}.$$



Zyro's tip

On my planet we have a saying: “no root left in the basement.” After every simplification, scan the denominator. If a radical is still hiding there, multiply by its conjugate and send it upstairs!

5. Solving radical equations

A **radical equation** has the unknown inside a radical. The idea is to undo the root by raising both sides to the matching power.



Method

1. Isolate the radical on one side of the equation.
2. Raise both sides to the power equal to the index.
3. Solve the resulting equation (linear or quadratic).
4. **Check every candidate** in the original equation.



Example 5

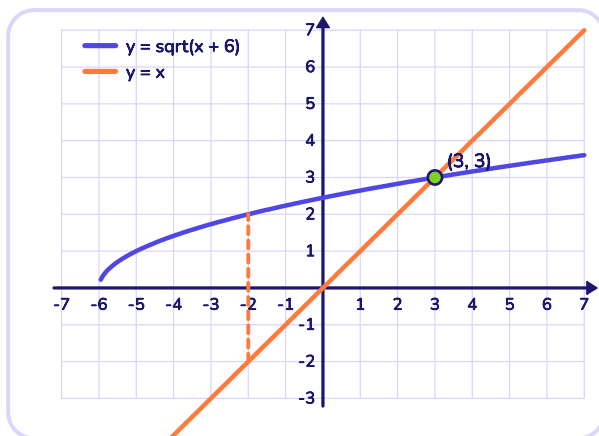
Solve $\sqrt{2x+1} + 3 = 8$, then $\sqrt[3]{x-4} = -2$.

Isolate: $\sqrt{2x+1} = 5$. Square both sides: $2x+1 = 25$, so $x = 12$. Check: $\sqrt{25} + 3 = 8$. It works.

Cube both sides: $x - 4 = -8$, so $x = -4$. Check: $\sqrt[3]{-8} = -2$. It works.

6. Extraneous solutions

Squaring is not reversible: $3^2 = (-3)^2$, yet $3 \neq -3$. So squaring an equation can create **extraneous solutions**: values that solve the new equation but not the original one. This happens with even indices; with odd indices it does not.



E Example 6

Solve $\sqrt{x+6} = x$.

Square both sides: $x+6 = x^2$, so $x^2 - x - 6 = 0$, which factors as $(x-3)(x+2) = 0$. The candidates are $x = 3$ and $x = -2$.

Check $x = 3$: $\sqrt{9} = 3$. True. Check $x = -2$: $\sqrt{4} = 2$, but the right side is -2 . False.

So $x = 3$ is the only solution and $x = -2$ is extraneous. The graph agrees: the curve and the line meet only at $(3, 3)$.



Never skip the check

A radical with an even index is never negative, so an equation like $\sqrt{x+6} = x$ forces $x \geq 0$. Checking the candidates in the *original* equation is the only reliable way to catch an extraneous solution.

7. Graphing radical functions

A radical function has the form $f(x) = a\sqrt[n]{x-h} + k$. It is the parent function $y = \sqrt[n]{x}$ after three changes:

- h slides the graph **right** (or left if h is negative);
- k slides it **up** (or down if k is negative);
- a stretches it vertically, and a negative a flips it over the horizontal line through its starting point.

For an even index, the graph begins at the **starting point** (h, k) . The domain is $x \geq h$, and the range is $y \geq k$ if $a > 0$ or $y \leq k$ if $a < 0$.

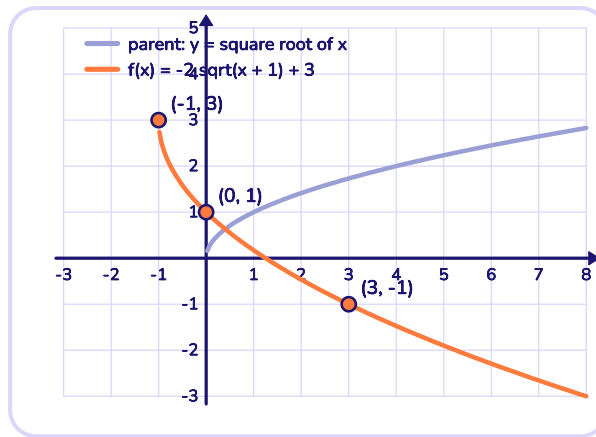


Example 7

Describe $f(x) = -2\sqrt{x+1} + 3$ and plot it.

Here $h = -1$, $k = 3$ and $a = -2$. The graph starts at $(-1, 3)$, then falls. The domain is $x \geq -1$ and the range is $y \leq 3$.

Points: $f(0) = -2 + 3 = 1$ and $f(3) = -2 \cdot 2 + 3 = -1$.



Key takeaways

- $\sqrt[n]{a} = a^{1/n}$ and $a^{m/n} = (\sqrt[n]{a})^m$: the denominator is the root, the numerator is the power.
- Even index: non-negative radicand and non-negative result. Odd index: any real number works.
- $\sqrt{x^2} = |x|$, and the usual exponent rules hold for rational exponents.
- Simplify by pulling out perfect powers, and rationalize denominators (conjugate for binomials).
- Solve a radical equation by isolating the radical, raising both sides to the index, and **checking** every answer.
- For $f(x) = a\sqrt{x-h} + k$, the graph starts at (h, k) with domain $x \geq h$.



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