

Polynomial Functions: math lesson, Grade 11 – download the PDF

Roller coaster hills, the profit of a growing company, the shape of a satellite dish: all of them can be modeled with **polynomial functions**. In this chapter you will learn to add, multiply and divide polynomials, to find their zeros with the Remainder, Factor and Rational Root Theorems, to sketch their graphs from end behavior, and to count every zero, real or complex, with the Fundamental Theorem of Algebra.

1. What is a polynomial function?

D

Polynomial function

A polynomial function in one variable has the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a whole number and $a_n \neq 0$. The number n is the **degree**, a_n is the **leading coefficient** and a_0 is the **constant term**. Writing the terms from the highest power to the lowest is called **standard form**.

Degree	Name	Example
0	constant	$f(x) = 7$
1	linear	$f(x) = 2x - 5$
2	quadratic	$f(x) = x^2 + 3x - 1$
3	cubic	$f(x) = x^3 - 4x$
4	quartic	$f(x) = -x^4 + 5x^2 - 4$
5	quintic	$f(x) = x^5 + 2x$



Watch out

The exponents must be whole numbers and the variable cannot be in a denominator or under a radical. So $\frac{4}{x} + x$ and $\sqrt{x} + 3$ are **not** polynomial functions.



Example 1: standard form

Write $f(x) = -3x^2 + 7x^5 - 4 + x$ in standard form and name its degree, leading coefficient and constant term.

Standard form: $f(x) = 7x^5 - 3x^2 + x - 4$. The degree is **5**, the leading coefficient is **7** and the constant term is **-4**.

2. Operations with polynomials

Adding and subtracting polynomials means combining **like terms**, the terms with the same power of x . Multiplying uses the distributive property: every term of the first factor multiplies every term of the second one. The sum, the difference and the product of two polynomials are always polynomials, and the degree of a product is the **sum** of the degrees of the factors.

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Example 2: subtracting

Compute $(5x^3 - 2x^2 + 7) - (3x^3 + x^2 - 4x + 1)$.

Distribute the minus sign: $5x^3 - 2x^2 + 7 - 3x^3 - x^2 + 4x - 1$. Combine like terms: $2x^3 - 3x^2 + 4x + 6$.

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Example 3: multiplying

Expand $(2x^2 - 3x + 5)(x - 4)$.

$2x^2(x - 4) = 2x^3 - 8x^2$, $-3x(x - 4) = -3x^2 + 12x$ and $5(x - 4) = 5x - 20$.

Adding gives $2x^3 - 11x^2 + 17x - 20$, a polynomial of degree $2 + 1 = 3$.



Watch out

When you subtract a polynomial, change the sign of **every** term inside the parentheses, not just the first one.

3. Polynomial long division

Dividing polynomials works like dividing whole numbers. You find a **quotient** $q(x)$ and a **remainder** $r(x)$ whose degree is smaller than the degree of the divisor $d(x)$, so that

$$p(x) = d(x)q(x) + r(x).$$

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Method: long division

1. Write both polynomials in standard form, with a **0 placeholder** for every missing power.
2. Divide the leading term of the dividend by the leading term of the divisor. This is the next term of the quotient.
3. Multiply the whole divisor by that term and subtract the result from the dividend.
4. Bring down the next term and repeat until the remainder has a smaller degree than the divisor.

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Example 4: long division

Divide $6x^3 - x^2 - 13x + 11$ by $2x - 1$.

$6x^3 \div 2x = 3x^2$. Then $3x^2(2x - 1) = 6x^3 - 3x^2$, and subtracting leaves $2x^2 - 13x$.

$2x^2 \div 2x = x$. Then $x(2x - 1) = 2x^2 - x$, and subtracting leaves $-12x + 11$.

$-12x \div 2x = -6$. Then $-6(2x - 1) = -12x + 6$, and subtracting leaves 5.

The quotient is $3x^2 + x - 6$ and the remainder is 5, so

$$6x^3 - x^2 - 13x + 11 = (2x - 1)(3x^2 + x - 6) + 5.$$

4. Synthetic division

When the divisor has the form $x - c$, there is a much faster shortcut that uses only the coefficients.

M

Method: synthetic division by $x - c$

1. Write c on the left and the coefficients of the dividend (with 0 placeholders) in a row.
2. Bring down the first coefficient.
3. Multiply it by c , write the product under the next coefficient and add.
4. Repeat. The last sum is the remainder; the other sums are the coefficients of the quotient, which has degree one less than the dividend.

$c = -2$	2	-3	0	5	-7
		-4	14	-28	46
	2	-7	14	-23	39

quotient $2x^3 - 7x^2 + 14x - 23$ rem. 39

E

Example 5: dividing by $x + 2$

Divide $2x^4 - 3x^3 + 5x - 7$ by $x + 2$.

Since $x + 2 = x - (-2)$, use $c = -2$. The dividend has no x^2 term, so its coefficients are $2, -3, 0, 5, -7$. The sums are $2, -7, 14, -23$ and the last one is 39 .

The quotient is $2x^3 - 7x^2 + 14x - 23$ and the remainder is 39 .

5. The Remainder and Factor Theorems

P Remainder Theorem

When a polynomial $p(x)$ is divided by $x - c$, the remainder is $p(c)$.

P Factor Theorem

$x - c$ is a factor of $p(x)$ if and only if $p(c) = 0$, that is, if and only if c is a zero of p .

The Remainder Theorem is a great way to evaluate a polynomial: instead of computing powers, run a synthetic division and read the remainder. In Example 5, $p(-2) = 39$, which you can confirm by direct substitution.

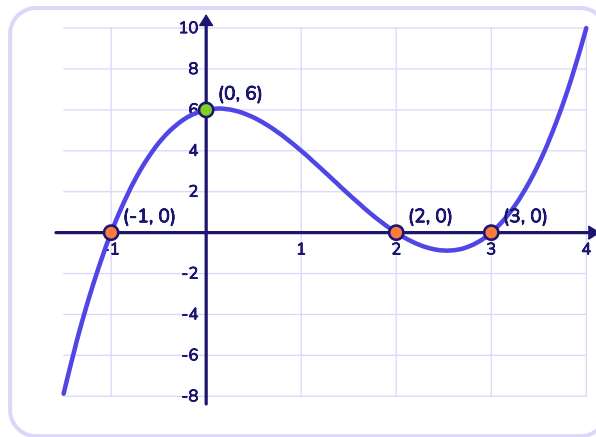
E Example 6: factoring a cubic

Show that $x - 3$ is a factor of $p(x) = x^3 - 4x^2 + x + 6$ and factor $p(x)$ completely.

$p(3) = 27 - 36 + 3 + 6 = 0$, so $x - 3$ is a factor. Synthetic division with $c = 3$ gives the sums 1, -1 , -2 and remainder 0, so

$$p(x) = (x - 3)(x^2 - x - 2) = (x - 3)(x - 2)(x + 1).$$

The zeros are 3, 2 and -1 , exactly where the graph below meets the x-axis.



E Example 7: using the remainder

What is the remainder when $4x^3 - 2x + 7$ is divided by $x + 1$?

Here $c = -1$, so the remainder is $p(-1) = -4 + 2 + 7 = 5$.

6. End behavior and graphs

For large values of $|x|$, a polynomial behaves like its leading term $a_n x^n$. That is why the degree and the sign of the leading coefficient tell you what the two ends of the graph do.

Degree	Leading coefficient	Left end ($x \rightarrow -\infty$)	Right end ($x \rightarrow +\infty$)
even	positive	rises	rises
even	negative	falls	falls
odd	positive	falls	rises
odd	negative	rises	falls

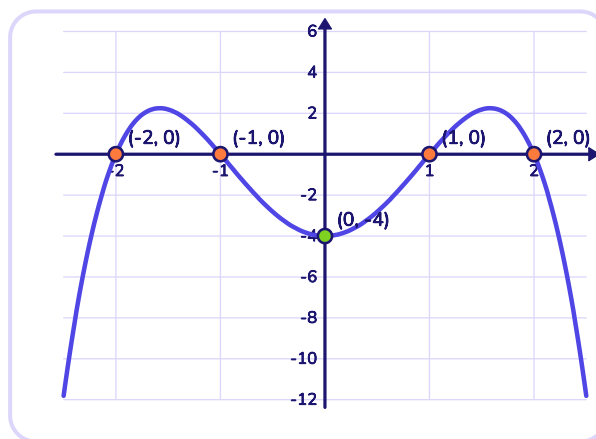
P**Zeros and turning points**

A polynomial function of degree n has at most n real zeros and at most $n - 1$ turning points. If $(x - c)$ appears m times in the factored form, then c is a zero of **multiplicity** m : the graph **crosses** the x-axis at c when m is odd and **touches** it and bounces back when m is even.

E**Example 8: sketching a quartic**

Describe $g(x) = -x^4 + 5x^2 - 4 = -(x - 1)(x + 1)(x - 2)(x + 2)$.

The degree is even and the leading coefficient is negative, so both ends fall. The zeros are ± 1 and ± 2 , each of multiplicity 1 , so the graph crosses the axis four times. The y-intercept is $g(0) = -4$, and there are at most 3 turning points.



7. The Rational Root Theorem

P Rational Root Theorem

Let $p(x)$ have integer coefficients. If $\frac{a}{b}$ is a rational zero of p written in lowest terms, then a divides the constant term and b divides the leading coefficient.

M Method: finding all zeros

1. List the divisors of the constant term (the possible a) and of the leading coefficient (the possible b). Form every fraction $\pm \frac{a}{b}$.
2. Test candidates with synthetic division until one gives remainder 0.
3. Rewrite $p(x)$ as (factor) $\times$ (quotient) and repeat on the quotient, or solve it with the quadratic formula.

E Example 9: all rational zeros

Solve $2x^3 - 3x^2 - 11x + 6 = 0$.

The divisors of 6 are $\pm 1, \pm 2, \pm 3, \pm 6$ and the divisors of 2 are $\pm 1, \pm 2$, so the candidates are $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$.

Try $x = 3$: $54 - 27 - 33 + 6 = 0$. Synthetic division gives the quotient $2x^2 + 3x - 2 = (2x - 1)(x + 2)$.

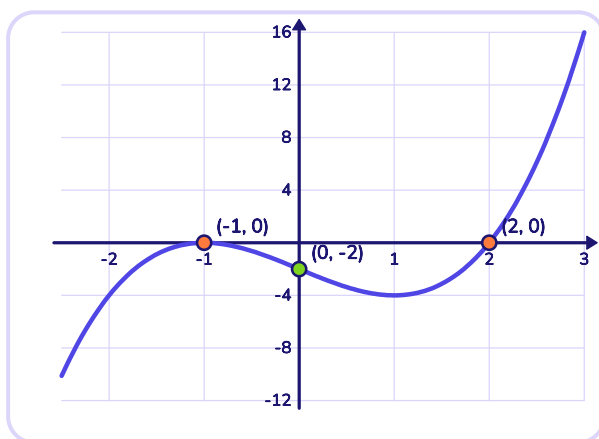
So $p(x) = (x - 3)(2x - 1)(x + 2)$ and the zeros are $3, \frac{1}{2}$ and -2 .

8. The Fundamental Theorem of Algebra

P Fundamental Theorem of Algebra

Every polynomial of degree $n \geq 1$ has at least one complex zero. Consequently, a polynomial of degree n has **exactly n complex zeros**, counted with multiplicity. If the coefficients are real, non-real zeros come in conjugate pairs $a + bi$ and $a - bi$.

Real numbers are complex numbers with imaginary part **0**, so the theorem counts every zero you can see on a graph and every one you cannot.



E Example 10: counting with multiplicity

The function $h(x) = (x + 1)^2(x - 2) = x^3 - 3x - 2$ has degree **3**, so it has three zeros: **-1** counted twice and **2**. The graph touches the axis at **-1** (even multiplicity) and crosses it at **2**.

E**Example 11: complex zeros**

Find all zeros of $x^3 - 3x^2 + 4x - 2$.

$p(1) = 1 - 3 + 4 - 2 = 0$, so $p(x) = (x - 1)(x^2 - 2x + 2)$. The quadratic formula gives $x = \frac{2 \pm \sqrt{4 - 8}}{2} = 1 \pm i$. The three zeros are $1, 1 + i$ and $1 - i$.

9. Sum and difference of cubes**P****Sum and difference of cubes**

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2) \quad a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

n	1	2	3	4	5	6	10
n^3	1	8	27	64	125	216	1,000

!**Zyro's trick**

Remember SOAP: the first sign is the **S**ame as in the original expression, the second is the **O**pposite, and the last is **A**lways **P**ositive.

E**Example 12: a sum of cubes**

Factor $27x^3 + 8$. Here $a = 3x$ and $b = 2$, so $27x^3 + 8 = (3x + 2)(9x^2 - 6x + 4)$.

E**Example 13: a difference of cubes**

Solve $x^3 - 125 = 0$. Factor: $(x - 5)(x^2 + 5x + 25) = 0$. The first factor gives $x = 5$. For the second, $x = \frac{-5 \pm \sqrt{25 - 100}}{2} = \frac{-5 \pm 5i\sqrt{3}}{2}$. That makes three zeros, as the Fundamental Theorem predicts.

**Watch out**

The quadratic factor $a^2 \mp ab + b^2$ does not factor further over the real numbers, and $(a + b)^3$ is **not** equal to $a^3 + b^3$.

Key takeaways

- The **degree** and the **leading coefficient** control the end behavior of a polynomial graph.
- Long division gives $p(x) = d(x)q(x) + r(x)$; synthetic division is the shortcut for divisors $x - c$.
- **Remainder Theorem:** $p(c)$ is the remainder on division by $x - c$. **Factor Theorem:** $p(c) = 0$ exactly when $x - c$ is a factor.
- **Rational Root Theorem:** rational zeros have the form $\pm \frac{\text{divisor of constant}}{\text{divisor of leading coefficient}}$.
- A degree- n polynomial has **exactly n complex zeros** with multiplicity; odd multiplicity means the graph crosses, even means it touches.
- $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$.



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