

Logarithmic Functions: math lesson, Grade 11 – download the PDF

Exponential functions grow fast, so a natural question is the reverse one: how long does it take? How many doublings turn 500 bacteria into 8,000? How many years does it take for savings to reach a goal? Logarithms answer exactly these questions. In this chapter you will learn what a logarithm is, how to work with it, how to graph it, and how to use it to solve exponential and logarithmic equations.

1. Logarithms as inverses of exponentials

D Definition of a logarithm

Let $b > 0$ with $b \neq 1$, and let $x > 0$. The **logarithm base b** of x is the exponent you put on b to get x :

$$\log_b x = y \iff b^y = x.$$

In words, $\log_b x$ answers the question “ b to what power gives x ?” For example, $2^5 = 32$, so $\log_2 32 = 5$. The two statements say the same thing, once in exponential form and once in logarithmic form.

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Example 1: evaluating logarithms

Evaluate (a) $\log_2 32$, (b) $\log_{10} 0.001$, (c) $\log_{1/3} 9$.

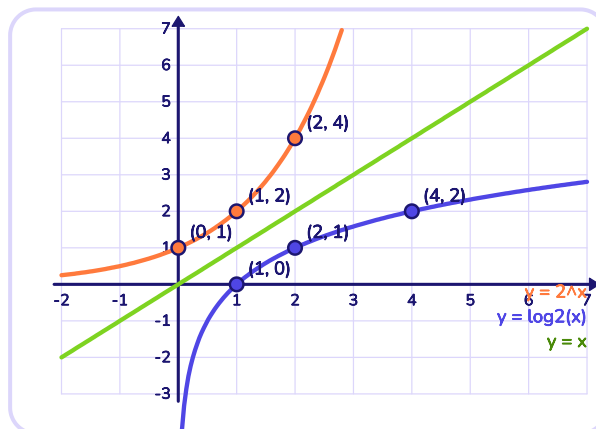
(a) $2^5 = 32$, so $\log_2 32 = 5$.

(b) $0.001 = 10^{-3}$, so $\log_{10} 0.001 = -3$.

(c) $\left(\frac{1}{3}\right)^{-2} = 3^2 = 9$, so $\log_{1/3} 9 = -2$.

A logarithm with base 10 is called a **common logarithm** and is written $\log x$ without a base. Your calculator's LOG key computes it.

Because $f(x) = b^x$ and $g(x) = \log_b x$ undo each other, they are **inverse functions**. Their graphs are mirror images across the line $y = x$: the point $(2, 4)$ on $y = 2^x$ becomes $(4, 2)$ on $y = \log_2 x$.



2. Switching forms and special values

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Method: convert between the two forms

1. Identify the base b : it is the small number next to “log”, or the number being raised to a power.
2. The logarithm (the value) is the exponent.
3. Rewrite: $\log_b x = y$ becomes $b^y = x$, and the other way around.

Because logarithms and exponentials undo each other, a few values always hold, for every valid base b :

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Special values and inverse identities

- $\log_b 1 = 0$, since $b^0 = 1$.
- $\log_b b = 1$, since $b^1 = b$.
- $\log_b (b^x) = x$ for every real x .
- $b^{\log_b x} = x$ for every $x > 0$.

⚠

Watch the domain

You cannot take the logarithm of zero or of a negative number: b^y is always positive. So $\log_3(-9)$ and $\log_5 0$ are **undefined**.

3. Properties of logarithms

Since logarithms are exponents, the exponent rules turn into logarithm rules.

P**Properties of logarithms**

For $b > 0$, $b \neq 1$, $M > 0$, $N > 0$ and any real number p :

- Product rule: $\log_b(MN) = \log_b M + \log_b N$
- Quotient rule: $\log_b \frac{M}{N} = \log_b M - \log_b N$
- Power rule: $\log_b(M^p) = p \log_b M$

Why does the product rule work? Write $M = b^u$ and $N = b^v$, so $u = \log_b M$ and $v = \log_b N$. Then $MN = b^{u+v}$, which means $\log_b(MN) = u + v$.

E**Example 2: expanding and condensing**

Expand $\log_5 \frac{25x^3}{\sqrt{y}}$.

$$\log_5 25 + \log_5 x^3 - \log_5 y^{1/2} = 2 + 3 \log_5 x - \frac{1}{2} \log_5 y.$$

Condense $2 \log_3 6 - \log_3 4$ into a single logarithm.

$$\log_3 6^2 - \log_3 4 = \log_3 \frac{36}{4} = \log_3 9 = 2.$$

⚠**Common mistakes**

$\log_b(M + N)$ is **not** $\log_b M + \log_b N$, and $\frac{\log_b M}{\log_b N}$ is **not** $\log_b M - \log_b N$. Also $(\log_b M)^2 \neq 2 \log_b M$: the power rule only works when the exponent is on the number inside.

4. Change of base

Most calculators only have the keys LOG (base 10) and LN (base e). The change-of-base formula lets you evaluate a logarithm in any base.

P Change-of-base formula

$$\log_b x = \frac{\log_a x}{\log_a b} = \frac{\ln x}{\ln b} \quad (a > 0, a \neq 1).$$

E Example 3: change of base

Approximate $\log_5 40$ to three decimal places.

$$\log_5 40 = \frac{\ln 40}{\ln 5} \approx \frac{3.6889}{1.6094} \approx 2.292.$$

Check: $5^{2.292} \approx 40$, and $5^2 = 25 < 40 < 125 = 5^3$, so a value between 2 and 3 makes sense.

5. Graphing logarithmic functions

The parent function $f(x) = \log_b x$ (with $b > 1$) has these features: its **domain** is $x > 0$, its **range** is all real numbers, it passes through $(1, 0)$ and $(b, 1)$, it is increasing, and the **y-axis** ($x = 0$) is a **vertical asymptote**.

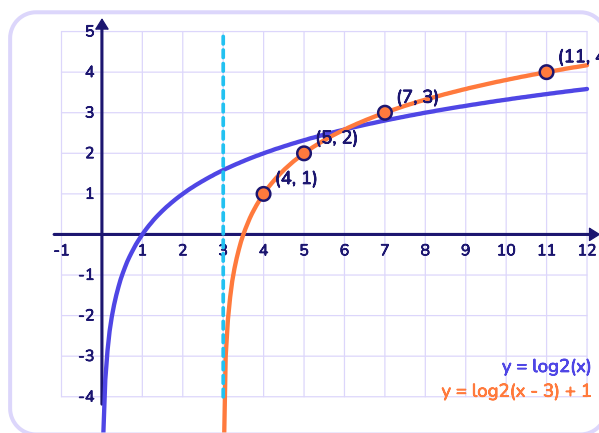
A general logarithmic function $g(x) = a \log_b(x - h) + k$ is the parent graph shifted h units right and k units up (and stretched or reflected by a). The vertical asymptote moves to $x = h$, and the domain becomes $x > h$.

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Example 4: graphing a translation

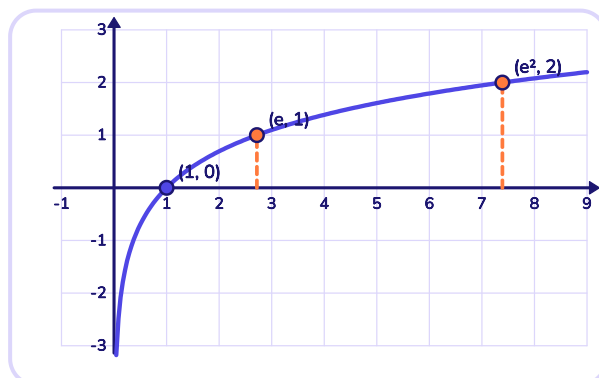
Graph $g(x) = \log_2(x - 3) + 1$.

The asymptote is $x = 3$ and the domain is $x > 3$. Take the parent points $(1, 0), (2, 1), (4, 2), (8, 3)$ and move each 3 units right and 1 unit up: $(4, 1), (5, 2), (7, 3), (11, 4)$. Draw a smooth curve through them that approaches the asymptote.



6. The natural logarithm

The number $e \approx 2.71828$ appears in continuous growth and decay. The **natural logarithm** is the logarithm with base e : $\ln x = \log_e x$. All the properties above still hold, and the special values become $\ln 1 = 0, \ln e = 1, \ln e^x = x$ and $e^{\ln x} = x$.



E Example 5: a natural logarithm

Solve $e^{0.4t} = 9$.

Take **ln** of both sides: $0.4t = \ln 9$, so $t = \frac{\ln 9}{0.4} \approx \frac{2.1972}{0.4} \approx 5.49$.

I 7. Solving exponential equations

M Method: solve an exponential equation

1. Isolate the power: get $b^{\text{expression}}$ alone on one side.
2. If both sides can be written with the same base, set the exponents equal.
3. Otherwise take a logarithm (usually **ln** or **log**) of both sides and use the power rule to bring the exponent down.
4. Solve for the variable and check with a calculator.

E Example 6: two exponential equations

(a) $3^{2x-1} = 81$. Since $81 = 3^4$, $2x - 1 = 4$, so $x = 2.5$.

(b) $1200(1.05)^t = 2000$. Divide: $(1.05)^t = \frac{5}{3}$. Then $t \ln 1.05 = \ln \frac{5}{3}$, so
 $t = \frac{\ln(5/3)}{\ln 1.05} \approx 10.47$ years.

I 8. Solving logarithmic equations

To solve a logarithmic equation, first combine the logs into one, then rewrite in exponential form.
Always check your answers: the number inside every logarithm must be positive, and a solution

that breaks this rule is called **extraneous**.

E Example 7: logarithmic equations

(a) $\log_3(x + 2) = 4$ gives $x + 2 = 3^4 = 81$, so $x = 79$.

(b) $\log_2 x + \log_2(x - 2) = 3$. Condense: $\log_2(x(x - 2)) = 3$, so $x^2 - 2x = 8$, that is $x^2 - 2x - 8 = (x - 4)(x + 2) = 0$. The candidates are 4 and -2 . But $\log_2(-2)$ is undefined, so $x = -2$ is extraneous. The only solution is $x = 4$.



Zyro's tip

On my planet we say: “a log is an exponent in disguise.” When you feel stuck, rewrite the logarithm as a power and the equation often becomes easy.

Key takeaways

- $\log_b x = y$ means $b^y = x$; it requires $b > 0$, $b \neq 1$ and $x > 0$.
- $\log_b 1 = 0$, $\log_b b = 1$, $\log_b b^x = x$ and $b^{\log_b x} = x$.
- Product, quotient and power rules turn products into sums, quotients into differences, and exponents into factors.
- Change of base: $\log_b x = \frac{\ln x}{\ln b}$.
- The graph of $\log_b x$ has domain $x > 0$, a vertical asymptote at $x = 0$, and passes through $(1, 0)$ and $(b, 1)$.
- Solve exponential equations with equal bases or by taking logs; solve log equations by rewriting as powers, then check for extraneous solutions.



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