

Limits, Probability, and Statistics: math lesson, Grade 12 – download the PDF

Some questions ask what a quantity *approaches*: the speed of a car at one exact instant, or the cost per item when a factory makes huge batches. Others ask how *likely* an event is, or how to summarize a pile of data. This chapter joins both worlds. You will study limits, count arrangements, compute probabilities with the binomial and normal models, and fit a line to real data.

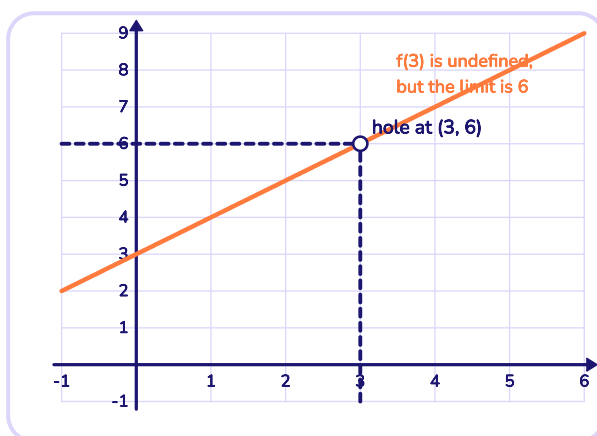
1. Intuitive limits and one-sided limits

D Limit

We write $\lim_{x \rightarrow a} f(x) = L$ when the values $f(x)$ get as close as we like to L as x gets close to a from *both* sides, with $x \neq a$. The value $f(a)$ itself does not matter, and it may not even exist.

E Example 1: a hole in the graph

Let $f(x) = \frac{x^2 - 9}{x - 3}$. We cannot plug in $x = 3$ because that gives $\frac{0}{0}$. Try nearby values: $f(2.9) = 5.9$, $f(2.99) = 5.99$, $f(3.01) = 6.01$, $f(3.1) = 6.1$. The outputs close in on 6. Algebra confirms it: for $x \neq 3$, $f(x) = \frac{(x - 3)(x + 3)}{x - 3} = x + 3$, so $\lim_{x \rightarrow 3} f(x) = 6$.



M Finding a limit at a number

1. Try direct substitution. If you get a real number, that is the limit (for a polynomial or a nice rational function).
2. If you get $\frac{0}{0}$, factor, simplify, or multiply by a conjugate, then substitute again.
3. If the function changes formula at a , look at each side separately.

D One-sided limits

$\lim_{x \rightarrow a^-} f(x)$ is the value approached when x comes from the **left** ($x < a$).

$\lim_{x \rightarrow a^+} f(x)$ is the value approached from the **right** ($x > a$).

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Two sides must agree

$\lim_{x \rightarrow a} f(x)$ exists if and only if both one-sided limits exist and are equal.

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Example 2: a jump

Let $f(x) = \begin{cases} 2x - 1 & \text{if } x < 1 \\ x^2 + 2 & \text{if } x \geq 1 \end{cases}$. From the left, $\lim_{x \rightarrow 1^-} f(x) = 2(1) - 1 = 1$. From the right, $\lim_{x \rightarrow 1^+} f(x) = 1^2 + 2 = 3$. Since $1 \neq 3$, the limit at $x = 1$ does not exist, even though $f(1) = 3$.

2. Limits at infinity

Here we ask what happens to $f(x)$ as x grows without bound. We write $\lim_{x \rightarrow \infty} f(x) = L$ when the graph levels off at the height L ; the line $y = L$ is a **horizontal asymptote**. Remember that $\frac{1}{x}$, $\frac{1}{x^2}$, and similar terms shrink to 0.

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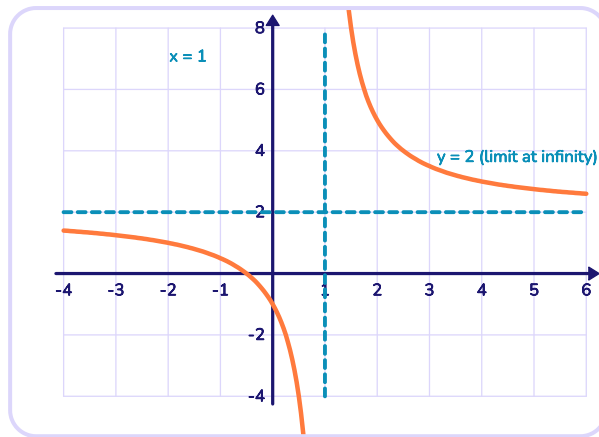
Rational functions at infinity

For $f(x) = \frac{a_n x^n + \dots}{b_m x^m + \dots}$: if $n < m$ the limit is 0; if $n = m$ the limit is $\frac{a_n}{b_m}$; if $n > m$ there is no finite limit (the values grow without bound).

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Example 3: dividing by the highest power

For $\lim_{x \rightarrow \infty} \frac{6x^2 - 5x}{2x^2 + 7}$, divide every term by x^2 : $\frac{6 - \frac{5}{x}}{2 + \frac{7}{x^2}} \rightarrow \frac{6}{2} = 3$. The line $y = 3$ is a horizontal asymptote.



The graph above shows $f(x) = \frac{2x+1}{x-1}$. Its horizontal asymptote is $y = 2$ because the degrees match and $\frac{2}{1} = 2$. It also has a **vertical asymptote** at $x = 1$, where the denominator is 0 and the numerator is not.



Careful

∞ is not a number. Writing $\lim_{x \rightarrow \infty} x^2 = \infty$ only describes how the values behave; it does not mean the limit exists as a real number.

3. Permutations and combinations

To count outcomes, use the **fundamental counting principle**: if one choice can be made in m ways and a second in n ways, the two together can be made in $m \cdot n$ ways. The factorial is $n! = n(n-1) \cdots 2 \cdot 1$, with $0! = 1$.

D

Permutations and combinations

A **permutation** is an arrangement where *order matters*. $P(n, r) = \frac{n!}{(n-r)!}$. A

combination is a selection where *order does not matter*. $C(n, r) = \frac{n!}{r!(n-r)!}$. They

are linked by $P(n, r) = r! \cdot C(n, r)$.

E**Example 4: medals or committees**

Eight sprinters race for gold, silver, and bronze. Order matters:

$P(8, 3) = 8 \cdot 7 \cdot 6 = 336$ podiums. A coach picks 4 of 10 students for a committee.

Order does not matter: $C(10, 4) = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4!} = 210$.

E**Example 5: repeated letters**

The word LEVEL has 5 letters, with L twice and E twice. The number of distinct

arrangements is $\frac{5!}{2! 2!} = \frac{120}{4} = 30$.

!**Order test**

Zyro asks: "If I swap two chosen items, do I get a different outcome?" If yes, use a permutation. If no, use a combination.

4. Conditional probability**D****Conditional probability**

The probability of A given that B has occurred is $P(A | B) = \frac{P(A \cap B)}{P(B)}$, for

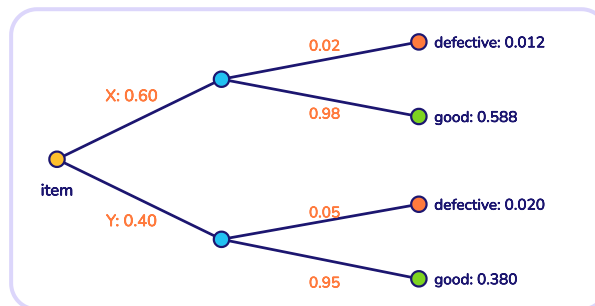
$P(B) > 0$. Rearranged, it gives the multiplication rule

$P(A \cap B) = P(B) \cdot P(A | B)$. Events are **independent** when $P(A | B) = P(A)$, that is, when $P(A \cap B) = P(A)P(B)$.

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Example 6: two machines

Machine X makes 60% of a factory's items and 2% of them are defective. Machine Y makes the other 40% and 5% of them are defective. Multiply along the branches of the tree:



Adding the two defective branches: $P(D) = 0.012 + 0.020 = 0.032$. Now suppose an item is found defective. The chance it came from Y is $P(Y | D) = \frac{0.020}{0.032} = 0.625$. So 62.5% of defective items come from Y, even though Y makes fewer items.



Careful

$P(A | B)$ and $P(B | A)$ are usually different. The chance that a defective item came from Y (0.625) is not the chance that a Y item is defective (0.05).

5. The binomial distribution

A random variable X is **binomial** when it counts successes in n trials that are **independent**, have only **two outcomes**, and share the same success probability p .

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Binomial formulas

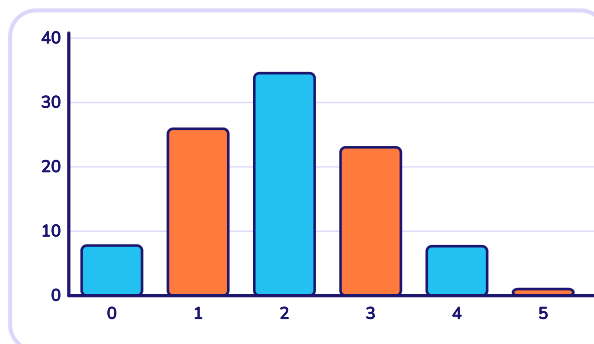
$P(X = k) = C(n, k) p^k (1 - p)^{n-k}$ for $k = 0, 1, \dots, n$. The mean is $\mu = np$ and the standard deviation is $\sigma = \sqrt{np(1 - p)}$.

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Example 7: free throws

A player makes 40% of her shots and takes 5 independent shots. Then

$P(X = 2) = C(5, 2)(0.4)^2(0.6)^3 = 10 \cdot 0.16 \cdot 0.216 = 0.3456$. The chance of at least one success is $1 - P(X = 0) = 1 - 0.6^5 = 0.92224$. The mean is $5 \cdot 0.4 = 2$ shots and $\sigma = \sqrt{5 \cdot 0.4 \cdot 0.6} = \sqrt{1.2} \approx 1.10$.



6. The normal distribution

Many measurements (heights, masses, test scores, manufacturing errors) pile up in a symmetric, bell-shaped curve centered at the mean μ . The spread is set by the standard deviation σ , and the total area under the curve is 1.

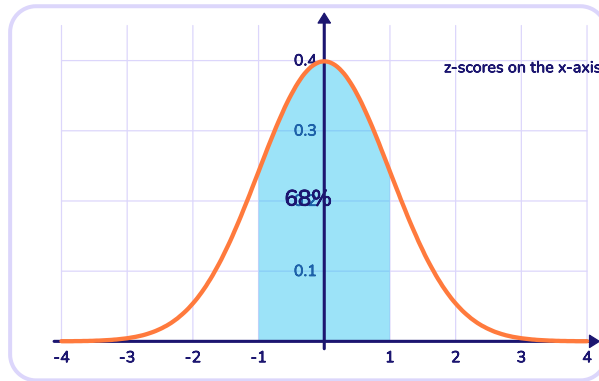
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Empirical rule (68-95-99.7)

In a normal distribution, about 68% of the values lie within 1σ of the mean, about 95% within 2σ , and about 99.7% within 3σ .

E**Example 8: apples**

Apple masses are normal with $\mu = 150$ g (about 5.3 oz) and $\sigma = 10$ g. About 68% weigh between 140 g and 160 g, and about 95% between 130 g and 170 g. The fraction above 170 g is $\frac{100\% - 95\%}{2} = 2.5\%$, because the curve is symmetric.

**7. Standard deviation and z-scores**

The **standard deviation** measures the typical distance between the data and their mean. For a whole population of n values, $\sigma = \sqrt{\frac{\sum (x - \mu)^2}{n}}$. For a sample, divide by $n - 1$ instead and write s .

E**Example 9: computing a standard deviation**

Data: 3, 5, 5, 7, 10, with mean $\mu = 6$.

x	$x - \mu$	$(x - \mu)^2$
3	-3	9
5	-1	1
5	-1	1
7	1	1
10	4	16

The sum of squares is 28. Population: $\sigma = \sqrt{28/5} = \sqrt{5.6} \approx 2.37$. Sample: $s = \sqrt{28/4} = \sqrt{7} \approx 2.65$.

D**z-score**

$z = \frac{x - \mu}{\sigma}$ tells how many standard deviations a value lies above ($z > 0$) or below ($z < 0$) the mean. It lets you compare values from different scales.

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Example 10: who did better?

Maya scored 84 on a test with $\mu = 72$ and $\sigma = 8$: $z = \frac{84 - 72}{8} = 1.5$. Leo scored 93 on a test with $\mu = 80$ and $\sigma = 10$: $z = 1.3$. Maya did better relative to her group.

8. Regression and correlation

When two quantities seem related, a **scatter plot** shows the pattern. The **least-squares line** $\hat{y} = mx + b$ is the line that makes the sum of squared vertical gaps as small as possible. With means $\bar{x}, \bar{y}$:

$$m = \frac{\sum(x - \bar{x})(y - \bar{y})}{\sum(x - \bar{x})^2}, \quad b = \bar{y} - m\bar{x}.$$

The **correlation coefficient** r , always between -1 and 1 , measures how close the points are to a line: r near 1 is a strong positive trend, near -1 a strong negative trend, near 0 no linear trend.

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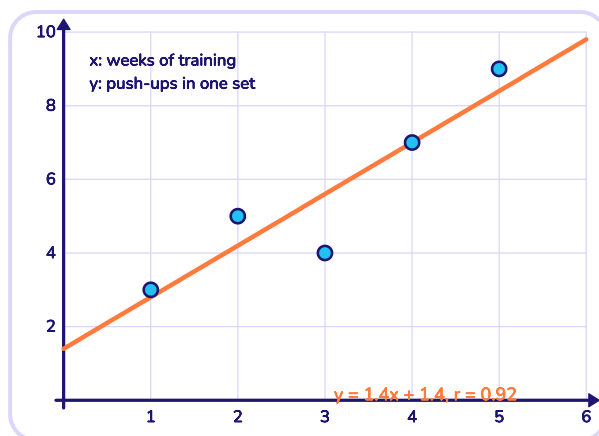
Example 11: training

After weeks of training $x = 1, 2, 3, 4, 5$, an athlete does $y = 3, 5, 4, 7, 9$ push-ups.

Here $\bar{x} = 3$, $\bar{y} = 5.6$, $\sum(x - \bar{x})(y - \bar{y}) = 14$, $\sum(x - \bar{x})^2 = 10$, and

$\sum(y - \bar{y})^2 = 23.2$. So $m = 1.4$, $b = 5.6 - 1.4 \cdot 3 = 1.4$, giving $\hat{y} = 1.4x + 1.4$,

and $r = \frac{14}{\sqrt{10 \cdot 23.2}} \approx 0.92$. Each extra week adds about 1.4 push-ups. At week 6 the model predicts 9.8.



Careful

A strong correlation does not prove that one variable *causes* the other. Also avoid **extrapolation**: predicting far outside the range of your data is unreliable.

Key takeaways

- A limit describes where $f(x)$ is heading; it ignores the value at the point. A two-sided limit exists only if both one-sided limits agree.
- For rational functions at infinity, compare the degrees of numerator and denominator.
- Order matters: $P(n, r) = \frac{n!}{(n-r)!}$. Order does not matter: $C(n, r) = \frac{n!}{r!(n-r)!}$.
- $P(A | B) = \frac{P(A \cap B)}{P(B)}$; a tree diagram multiplies along branches and adds across branches.
- Binomial: $P(X = k) = C(n, k)p^k(1-p)^{n-k}$, $\mu = np$, $\sigma = \sqrt{np(1-p)}$.
- Normal data: about 68%, 95%, and 99.7% lie within 1, 2, and 3 standard deviations;
 $z = \frac{x - \mu}{\sigma}$.
- The least-squares line has $m = \frac{\sum(x - \bar{x})(y - \bar{y})}{\sum(x - \bar{x})^2}$; r near ± 1 means a strong linear trend, but never proves causation.



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