

Limits and Continuity: math lesson, College – download the PDF

How close can a function get to a value without ever touching it? Limits answer exactly that question, and they are the foundation of everything in calculus: derivatives, integrals, and infinite series all rest on them. In this chapter you will learn to read limits from tables and graphs, compute them with algebra, understand what happens at infinity, and use continuity to guarantee that a function has no surprises.

1. The intuitive idea of a limit

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Definition (limit)

Let f be a function defined near a (it may be undefined at a itself). We write $\lim_{x \rightarrow a} f(x) = L$ when the values $f(x)$ get as close to L as we like by taking x close enough to a , with $x \neq a$.

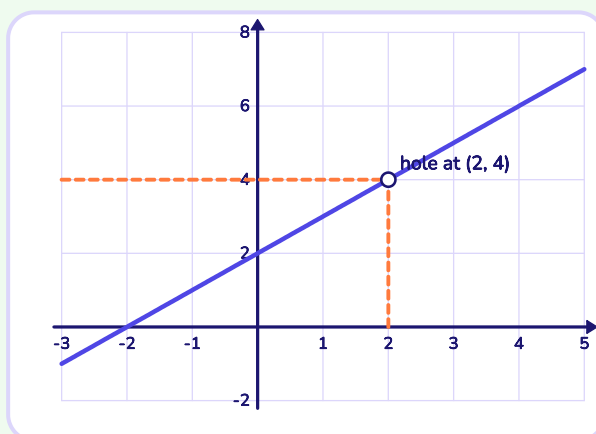
The key phrase is “with $x \neq a$ ”. A limit describes the *behavior near* a , not the value *at* a . The function can even be undefined at a , as in the next example.

E Example 1: a hole in the graph

Let $f(x) = \frac{x^2 - 4}{x - 2}$. It is undefined at $x = 2$, but for $x \neq 2$ we can factor:
 $f(x) = \frac{(x - 2)(x + 2)}{x - 2} = x + 2$. Check numerically:

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999	4.001	4.01	4.1

The values approach 4 from both sides, so $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = 4$. The graph is the line $y = x + 2$ with one point removed.



Zyro says A table of values gives you a good *guess*, never a proof. Use it to predict the limit, then confirm it with algebra.

2. One-sided limits

Sometimes the function behaves differently on each side of a . We write $\lim_{x \rightarrow a^-} f(x)$ for the **left-hand limit** (using only $x < a$) and $\lim_{x \rightarrow a^+} f(x)$ for the **right-hand limit** (using only $x > a$).

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Property (two sides)

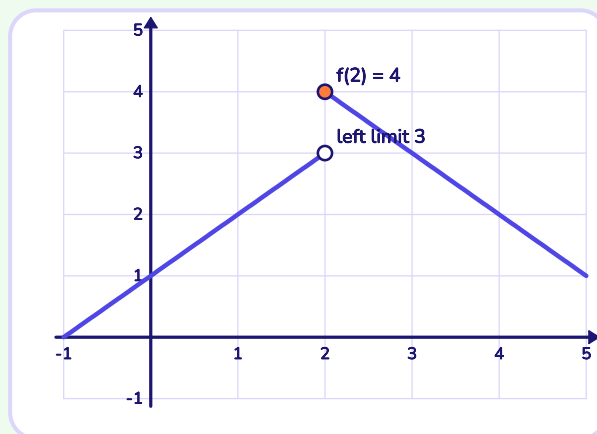
$\lim_{x \rightarrow a} f(x) = L$ if and only if $\lim_{x \rightarrow a^-} f(x) = L$ and $\lim_{x \rightarrow a^+} f(x) = L$. If the two one-sided limits differ, the limit does not exist.

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Example 2: a jump

Let $f(x) = x + 1$ for $x < 2$ and $f(x) = 6 - x$ for $x \geq 2$. Then

$\lim_{x \rightarrow 2^-} f(x) = 2 + 1 = 3$ and $\lim_{x \rightarrow 2^+} f(x) = 6 - 2 = 4$. Since $3 \neq 4$, $\lim_{x \rightarrow 2} f(x)$ does not exist, even though $f(2) = 4$.



One-sided limits also describe blow-up. For x slightly bigger than 0, $\frac{1}{x}$ is a huge positive number, so $\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$; on the left it is hugely negative, so $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$. The symbol ∞ tells you how the function behaves, but it is not a real number, so such a limit is said to be infinite (and does not exist as a finite number).

3. Limit laws

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Limit laws

Suppose $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$ are both finite numbers, and let c be a constant. Then:

- $\lim(f + g) = L + M$ and $\lim(f - g) = L - M$
- $\lim cf = cL$ and $\lim fg = LM$
- $\lim \frac{f}{g} = \frac{L}{M}$ provided $M \neq 0$
- $\lim (f(x))^n = L^n$ and $\lim \sqrt[n]{f(x)} = \sqrt[n]{L}$ (when the root is defined)

For polynomials and many other functions, **direct substitution** works: $\lim_{x \rightarrow a} f(x) = f(a)$. For example, $\lim_{x \rightarrow 3} (2x^2 - x) = 18 - 3 = 15$. Trouble appears when substitution gives the indeterminate form $\frac{0}{0}$. Then you must rewrite the expression first.

M

Method: handling $\frac{0}{0}$

1. Substitute $x = a$. If you get a nonzero number, you are done.
2. If you get $\frac{0}{0}$, factor the numerator and denominator and cancel the common factor $(x - a)$; or, if a square root is involved, multiply top and bottom by the conjugate.
3. Substitute again in the simplified expression.

E**Example 3: using a conjugate**

Find $\lim_{x \rightarrow 3} \frac{\sqrt{x+1}-2}{x-3}$. Substitution gives $\frac{0}{0}$. Multiply by the conjugate $\sqrt{x+1}+2$:

$$\frac{\sqrt{x+1}-2}{x-3} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \frac{(x+1)-4}{(x-3)(\sqrt{x+1}+2)} = \frac{1}{\sqrt{x+1}+2}.$$

As $x \rightarrow 3$, this tends to $\frac{1}{2+2} = \frac{1}{4}$.

4. Limits at infinity and asymptotes

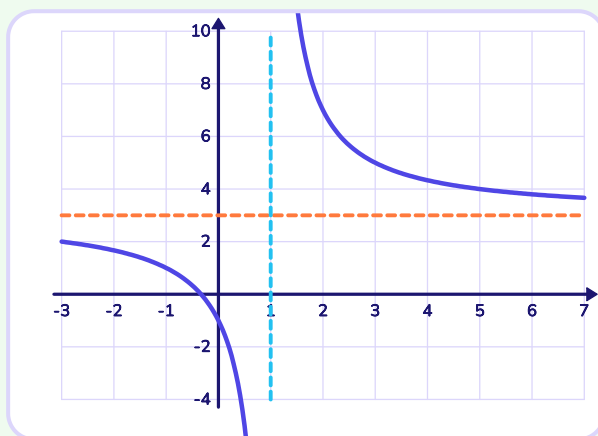
We also ask what happens when x becomes very large: $\lim_{x \rightarrow +\infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$. The basic fact is $\lim_{x \rightarrow \pm\infty} \frac{1}{x^n} = 0$ for every positive integer n .

M**Method: rational functions at infinity**

Divide the numerator and the denominator by the highest power of x that appears in the denominator, then use $\frac{1}{x^n} \rightarrow 0$.

E Example 4: horizontal and vertical asymptotes

For $g(x) = \frac{3x+1}{x-1}$, divide by x : $g(x) = \frac{3 + \frac{1}{x}}{1 - \frac{1}{x}} \rightarrow \frac{3}{1} = 3$ as $x \rightarrow \pm\infty$. So the line $y = 3$ is a **horizontal asymptote**. At $x = 1$ the denominator is 0 while the numerator is 4, so g blows up: $x = 1$ is a **vertical asymptote**.



D Asymptotes

The line $y = L$ is a horizontal asymptote if $\lim_{x \rightarrow +\infty} f(x) = L$ or $\lim_{x \rightarrow -\infty} f(x) = L$. The line $x = a$ is a vertical asymptote if at least one one-sided limit at a is $+\infty$ or $-\infty$.



Watch out When you take roots, signs matter: $\sqrt{x^2} = |x|$, which equals $-x$ when x is negative. The limits at $+\infty$ and $-\infty$ of $\frac{\sqrt{x^2}}{x}$ are 1 and -1 .

5. The squeeze theorem

P Squeeze theorem

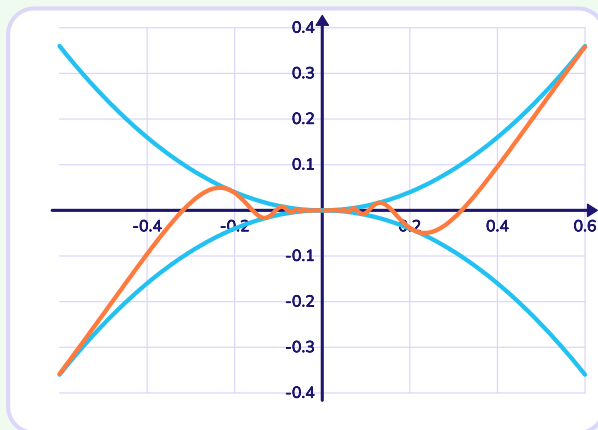
If $h(x) \leq f(x) \leq g(x)$ for all x near a (except possibly at a) and $\lim_{x \rightarrow a} h(x) = \lim_{x \rightarrow a} g(x) = L$, then $\lim_{x \rightarrow a} f(x) = L$. The same statement holds as $x \rightarrow \pm\infty$.

E Example 5: an oscillating function

Find $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x}$. The function $\sin \frac{1}{x}$ oscillates wildly and has no limit at 0, but it always lies between -1 and 1 . Multiplying by $x^2 \geq 0$:

$$-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2.$$

Both $-x^2$ and x^2 tend to 0, so the middle function also tends to 0.



6. Continuity at a point

D Definition (continuity)

A function f is **continuous at a** if three conditions hold:

1. $f(a)$ is defined;
2. $\lim_{x \rightarrow a} f(x)$ exists (as a finite number);
3. $\lim_{x \rightarrow a} f(x) = f(a)$.

Informally, you can draw the graph near a without lifting your pencil. Polynomials, $\sin x$, $\cos x$, e^x , and roots are continuous wherever they are defined, and sums, products, quotients (with nonzero denominator), and compositions of continuous functions are continuous.

E Example 6: choosing a value to repair a function

Let $f(x) = \frac{x^2 - 9}{x - 3}$ for $x \neq 3$. For which value $k = f(3)$ is f continuous at 3? We have $f(x) = x + 3$ for $x \neq 3$, so $\lim_{x \rightarrow 3} f(x) = 6$. Continuity requires $f(3) = \lim_{x \rightarrow 3} f(x)$, so $k = 6$.

7. Types of discontinuity

When one of the three conditions fails, f is discontinuous at a . There are three classic shapes.

- **Removable**: the limit exists but $f(a)$ is undefined or different from the limit (a hole). You can repair it by redefining $f(a)$.
- **Jump**: both one-sided limits exist but are different numbers.
- **Infinite**: at least one one-sided limit is $\pm\infty$ (a vertical asymptote).



Common mistake “ $f(a)$ exists, so the limit exists.” False! The value at the point and the limit near the point are separate questions. Always check all three conditions.

8. The Intermediate Value Theorem



Intermediate Value Theorem (NT)

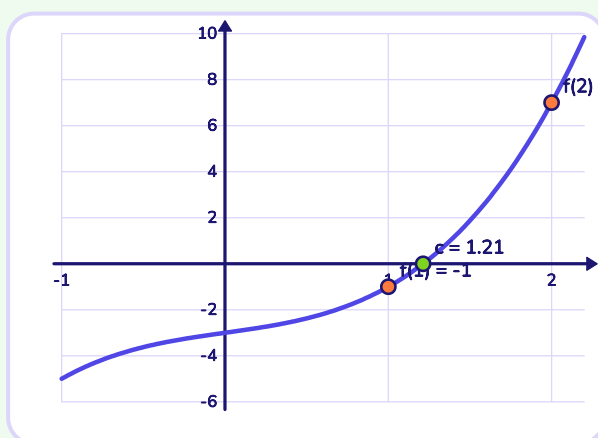
If f is continuous on the closed interval $[a, b]$ and N is any number between $f(a)$ and $f(b)$, then there is at least one c in $[a, b]$ with $f(c) = N$.

The theorem is most often used to show that an equation has a solution: if f is continuous and $f(a)$ and $f(b)$ have opposite signs, then f has a root between a and b . Continuity is essential; a function with a jump can skip values.



Example 7: locating a root

Show that $x^3 + x - 3 = 0$ has a solution in $(1, 2)$. Let $f(x) = x^3 + x - 3$, a polynomial and therefore continuous. We have $f(1) = 1 + 1 - 3 = -1 < 0$ and $f(2) = 8 + 2 - 3 = 7 > 0$. Since 0 lies between -1 and 7 , the IVT gives some $c \in (1, 2)$ with $f(c) = 0$.



Key takeaways

- $\lim_{x \rightarrow a} f(x) = L$ describes what $f(x)$ approaches as x nears a ; it ignores $f(a)$.
- The limit exists exactly when the left-hand and right-hand limits exist and are equal.
- For a $\frac{0}{0}$ form, factor and cancel, or use a conjugate, then substitute.
- At infinity, divide by the highest power of x ; $\frac{1}{x^n} \rightarrow 0$. Horizontal asymptotes come from limits at $\pm\infty$, vertical ones from infinite one-sided limits.
- Squeeze theorem: a function trapped between two functions with the same limit has that limit.
- Continuity at a : $f(a)$ defined, limit exists, and they are equal.
- Discontinuities are removable, jump, or infinite.
- IVT: continuous on $[a, b]$ means every value between $f(a)$ and $f(b)$ is reached.



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