

Law of Sines and Law of Cosines: math lesson, Grade 12 – download the PDF

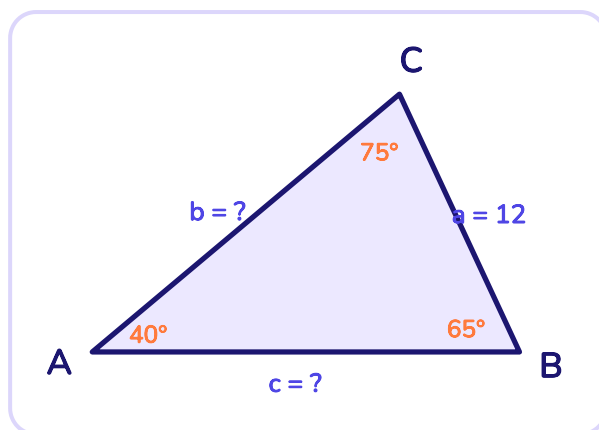
Right triangles are easy: SOH-CAH-TOA does the job. But most triangles you meet in surveying, navigation and engineering have *no* right angle. This chapter gives you two powerful tools, the Law of Sines and the Law of Cosines, that let you solve *any* triangle, find its area, and answer questions about bearings and distances.

1. Oblique triangles and the naming convention

D Oblique triangle

An oblique triangle is a triangle with no right angle. To *solve* a triangle means to find all three sides and all three angles.

To solve one, you need three well-chosen measurements. Throughout the chapter we use one convention: **side a is opposite angle A** , side **b** is opposite angle **B** , and side **c** is opposite angle **C** . Angles add up to 180° , and the longest side always faces the largest angle.



The possible “given” patterns are named by the order in which the known parts appear around the triangle: AAS or ASA (two angles and a side), SAS (two sides and the angle between them), SSS (three sides) and SSA (two sides and an angle *not* between them). Each pattern points to a different strategy, as you will see in Part 5.

2. The Law of Sines

P Law of Sines

In every triangle ABC ,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

This common value equals $2R$, where R is the radius of the circle passing through the three vertices.

Why it works. Draw the altitude h from C to side AB . It gives $h = b \sin A$ and also $h = a \sin B$.

So $b \sin A = a \sin B$, which rearranges to $\frac{a}{\sin A} = \frac{b}{\sin B}$. Repeating with another altitude brings in c .

E

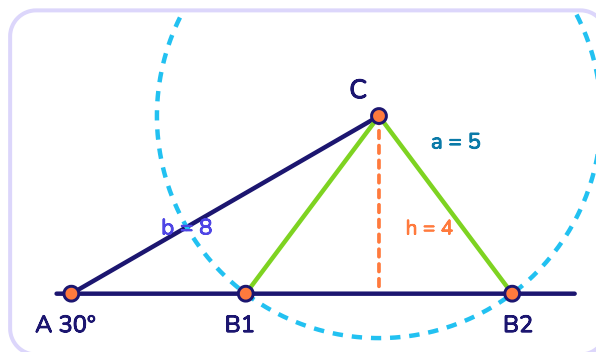
Example 1: AAS

In triangle ABC , $A = 40^\circ$, $B = 65^\circ$ and $a = 12$ cm. Find C , b and c .

First, $C = 180^\circ - 40^\circ - 65^\circ = 75^\circ$. Then $b = \frac{a \sin B}{\sin A} = \frac{12 \sin 65^\circ}{\sin 40^\circ} \approx 16.92$ cm
and $c = \frac{12 \sin 75^\circ}{\sin 40^\circ} \approx 18.03$ cm.

3. The ambiguous case (SSA)

When you know two sides and an angle that is *not* between them, the Law of Sines gives $\sin B$, and an angle with a given sine can be acute *or* obtuse: B and $180^\circ - B$ share the same sine. Sometimes both choices produce a real triangle, sometimes only one, sometimes none.



M

Method: sorting out the SSA case

1. Compute $\sin B = \frac{b \sin A}{a}$. If the result is greater than 1, **no triangle** exists.
2. If $\sin B \leq 1$, find the acute angle $B_1 = \sin^{-1}(\sin B)$ and the obtuse candidate $B_2 = 180^\circ - B_1$.
3. Keep a candidate only if $A + B < 180^\circ$. Each valid B gives $C = 180^\circ - A - B$ and then $c = \frac{a \sin C}{\sin A}$.

With A acute and $h = b \sin A$, the outcomes are summarized below.

Condition	Number of triangles
$a < h$	None
$a = h$	One (right triangle)
$a > h$	Two
$a \geq b$	One

If A is right or obtuse, there is exactly one triangle when $a > b$ and none otherwise.

E Example 2: two triangles

Let $A = 30^\circ$, $a = 5$ and $b = 8$. Here $h = 8 \sin 30^\circ = 4 < 5 < 8$, so expect two triangles. We get $\sin B = \frac{8 \sin 30^\circ}{5} = 0.8$, so $B_1 \approx 53.13^\circ$ and $B_2 \approx 126.87^\circ$. Both satisfy $A + B < 180^\circ$.

Triangle 1: $C_1 \approx 96.87^\circ$, $c_1 = \frac{5 \sin 96.87^\circ}{\sin 30^\circ} \approx 9.93$. Triangle 2: $C_2 \approx 23.13^\circ$, $c_2 \approx 3.93$.

4. The Law of Cosines

P Law of Cosines

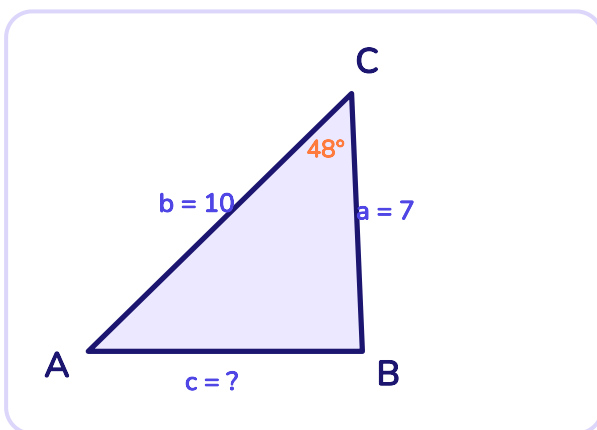
In every triangle ABC ,

$$c^2 = a^2 + b^2 - 2ab \cos C,$$

and likewise $a^2 = b^2 + c^2 - 2bc \cos A$ and $b^2 = a^2 + c^2 - 2ac \cos B$. Solved for an angle:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

When $C = 90^\circ$, $\cos C = 0$ and the formula collapses to the Pythagorean theorem, so the Law of Cosines is its generalization. If C is obtuse, $\cos C < 0$, so $c^2 > a^2 + b^2$.



E Example 3: SAS

Let $a = 7$, $b = 10$ and $C = 48^\circ$. Then $c^2 = 49 + 100 - 140 \cos 48^\circ \approx 55.32$, so $c \approx 7.44$. To find A , use the Law of Sines (safe here, because a is the shortest side so A is acute): $\sin A = \frac{7 \sin 48^\circ}{7.44}$, giving $A \approx 44.4^\circ$ and $B \approx 87.6^\circ$.

E Example 4: SSS

Sides **6**, **8** and **11**. Start with the largest angle, opposite **11**:

$\cos C = \frac{36 + 64 - 121}{2 \cdot 6 \cdot 8} = -\frac{21}{96}$, so $C \approx 102.6^\circ$, an obtuse angle. Starting with the largest angle is smart: after that, the other two angles must be acute, so the Law of Sines can finish the job without ambiguity.

5. Choosing the right law

Given	First step
AAS or ASA	Find the third angle, then Law of Sines
SAS	Law of Cosines for the third side
SSS	Law of Cosines for the largest angle
SSA	Law of Sines, then check for 0, 1 or 2 triangles

Common mistake

Do not use the Law of Sines to find an angle that might be obtuse unless you have checked the alternative $180^\circ - B$. The cosine formula avoids the problem, because cosine is negative for obtuse angles and distinguishes them.

6. Area of a triangle using sine

P Area formula (SAS)

The area of a triangle with sides a and b enclosing angle C is

$$K = \frac{1}{2} ab \sin C.$$

This is just $\frac{1}{2} \times \text{base} \times \text{height}$ with height $h = b \sin C$. The same formula works with any pair of sides and the angle between them.

E Example 5

A triangular garden bed has sides 9 ft and 14 ft meeting at 35° . Its area is

$$K = \frac{1}{2}(9)(14) \sin 35^\circ \approx 36.14 \text{ ft}^2, \text{ which is about } 3.36 \text{ m}^2.$$

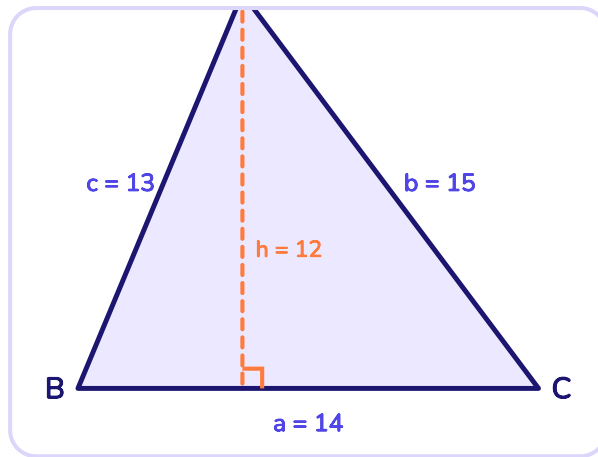
7. Heron's formula

When you know all three sides and no angle, you can skip the cosine entirely.

P Heron's formula

Let $s = \frac{a + b + c}{2}$ be the semiperimeter. Then

$$K = \sqrt{s(s - a)(s - b)(s - c)}.$$

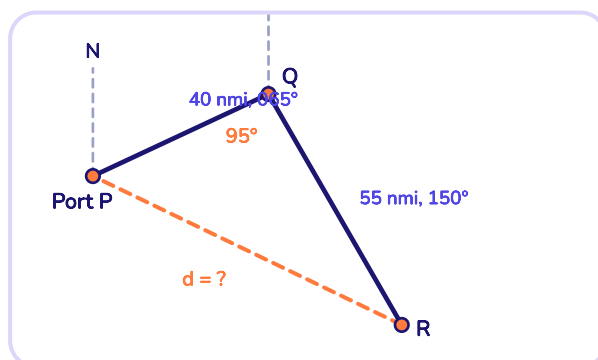


E Example 6

For sides 13, 14 and 15: $s = 21$, so $K = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = \sqrt{7056} = 84$. Check with the figure: $\frac{1}{2}(14)(12) = 84$.

8. Bearings and navigation

In navigation, a **bearing** is an angle measured *clockwise from north* and written with three digits, such as **065°** (a little north of east) or **210°** (south-southwest). The key to bearing problems: draw a north line at every turning point, then compute the interior angle of the triangle from the bearings.



E Example 7: a two-leg voyage

A ship leaves port P and sails 40 nautical miles (nmi, about 74 km) on bearing 065° to point Q, then 55 nmi on bearing 150° to point R. How far is R from the port?

At Q, the direction back to P has bearing $065^\circ + 180^\circ = 245^\circ$. The new heading is 150° , so the interior angle at Q is $245^\circ - 150^\circ = 95^\circ$. By the Law of Cosines, $d^2 = 40^2 + 55^2 - 2(40)(55) \cos 95^\circ \approx 5008.5$, so $d \approx 70.8$ nmi (about 131 km).

! Zyro's tip

On my home planet we navigate by two moons, and we always sketch first! A rough north-arrow diagram at each turn catches most sign errors before they happen.

Key takeaways

- Law of Sines: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$. Use it with AAS, ASA, or SSA.
- SSA can give 0, 1 or 2 triangles: compare a with $h = b \sin A$ and with b .
- Law of Cosines: $c^2 = a^2 + b^2 - 2ab \cos C$. Use it with SAS or SSS, and find the largest angle first.
- Area: $K = \frac{1}{2}ab \sin C$, or Heron's $K = \sqrt{s(s-a)(s-b)(s-c)}$ when only sides are known.
- Bearings are measured clockwise from north; draw a north line at every turn.



Test yourself: quick challenge for Grade 12

[Change level](#)

Level : **Grade 12**

6 random questions, instant feedback.



Start the challenge



Speed drill for Grade 12: how many in 60 seconds?



Keep exploring with Zyro



Math practice

Law of Sines and Law of Cosines: math practice, Grade 12



Math tests

Law of Sines and Law of Cosines: math test, Grade 12

 Math quizzes

Law of Sines and Law of Cosines: math quiz, Grade 12

 Math practice

Vectors, Polar Coordinates, and Complex Numbers: math practice, Grade 12

 Math practice

Conic Sections and Parametric Equations: math practice, Grade 12

 Math tests

Vectors, Polar Coordinates, and Complex Numbers: math test, Grade 12

Zyro takes you to our partner sites

Blast off to other planets of knowledge and their apps.

Lift off