

Techniques of Integration: practice solutions, College – download the PDF

Written solutions to the chapter problems. Check each step, then correct yourself.

1 Parts with cosine



Take $u = x$, $dv = \cos x \, dx$, so $du = dx$, $v = \sin x$.

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C.$$

2 Parts with sin 2x



Let $u = x$, $dv = \sin 2x \, dx$; then $du = dx$, $v = -\frac{1}{2} \cos 2x$.

$$\int x \sin 2x \, dx = -\frac{x}{2} \cos 2x + \frac{1}{2} \int \cos 2x \, dx = -\frac{x}{2} \cos 2x + \frac{1}{4} \sin 2x + C.$$

3 Odd power of cosine



Write $\cos^3 x = (1 - \sin^2 x) \cos x$ and let $u = \sin x$, $du = \cos x \, dx$.

$$\int (1 - u^2) du = u - \frac{u^3}{3} + C = \sin x - \frac{\sin^3 x}{3} + C.$$

4 Sine times a fourth power



Let $u = \cos x$, so $du = -\sin x \, dx$. Then $-\int u^4 du = -\frac{u^5}{5} + C = -\frac{\cos^5 x}{5} + C.$

5 Simple partial fractions



$\frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$ with $1 = A(x+2) + B(x-2)$. At $x = 2$: $A = \frac{1}{4}$; at $x = -2$: $B = -\frac{1}{4}$.

$$\int \frac{dx}{x^2 - 4} = \frac{1}{4} \ln|x-2| - \frac{1}{4} \ln|x+2| + C = \frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| + C.$$

6 A first improper integral



$\int_1^b x^{-3} dx = \left[-\frac{1}{2x^2} \right]_1^b = \frac{1}{2} - \frac{1}{2b^2}$. As $b \rightarrow \infty$ this tends to $\frac{1}{2}$. The integral converges to $\frac{1}{2}$ (p-test with $p = 3$).

7 Pick the technique



- a. u -substitution with $u = x^2$, because $2x dx = du$ is already present.
- b. Integration by parts: a polynomial times an exponential.
- c. Partial fractions: the denominator factors as $(x-3)(x+3)$.
- d. The half-angle identity $\sin^2 x = \frac{1-\cos 2x}{2}$.

8 Double parts



First $u = x^2$, $dv = e^x dx$: $\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx$. Second, $\int x e^x dx = (x-1)e^x$. So an antiderivative is $e^x(x^2 - 2x + 2)$.

$$\int_0^1 = e(1 - 2 + 2) - (1)(2) = e - 2 \approx 0.718.$$

9 Root times logarithm



Take $u = \ln x$, $dv = x^{1/2}dx$, $v = \frac{2}{3}x^{3/2}$. Then

$$\int \sqrt{x} \ln x \, dx = \frac{2}{3}x^{3/2} \ln x - \frac{2}{3} \int x^{1/2} dx = \frac{2}{3}x^{3/2} \ln x - \frac{4}{9}x^{3/2}.$$

At 4: $\frac{16}{3} \ln 4 - \frac{32}{9}$. At 1: $-\frac{4}{9}$. Difference: $\frac{16}{3} \ln 4 - \frac{28}{9} \approx 4.282$.

10 Even powers



$\sin x \cos x = \frac{1}{2} \sin 2x$, so $\sin^2 x \cos^2 x = \frac{1}{4} \sin^2 2x = \frac{1 - \cos 4x}{8}$.

$$\int = \frac{x}{8} - \frac{\sin 4x}{32} + C.$$

11 Area of a quarter disk



Let $x = 2 \sin \theta$, $dx = 2 \cos \theta \, d\theta$; $x = 0 \rightarrow \theta = 0$, $x = 2 \rightarrow \theta = \frac{\pi}{2}$. The integrand is $2 \cos \theta$, so the integral is $\int_0^{\pi/2} 4 \cos^2 \theta \, d\theta = 2 \left[\theta + \sin \theta \cos \theta \right]_0^{\pi/2} = \pi$.

Geometrically this is the area of a quarter of a disk of radius 2: $\frac{1}{4} \pi \cdot 2^2 = \pi$.

12 Two linear factors



$x^2 + x - 2 = (x - 1)(x + 2)$. Write $2x + 3 = A(x + 2) + B(x - 1)$. At $x = 1$: $5 = 3A$, $A = \frac{5}{3}$. At $x = -2$: $-1 = -3B$, $B = \frac{1}{3}$.

$$\int = \frac{5}{3} \ln |x - 1| + \frac{1}{3} \ln |x + 2| + C.$$

13 An exponential tail



By parts, $u = x$, $dv = e^{-3x} dx$: $\int x e^{-3x} dx = -\frac{x}{3} e^{-3x} - \frac{1}{9} e^{-3x}$. At $b \rightarrow \infty$ both terms tend to 0; at 0 the value is $-\frac{1}{9}$.

So the integral equals $0 - (-\frac{1}{9}) = \frac{1}{9}$.

14 Integral of $\ln x$ near zero



Antiderivative: $x \ln x - x$. So the integral is

$\lim_{t \rightarrow 0^+} [x \ln x - x]_t^1 = (0 - 1) - \lim_{t \rightarrow 0^+} (t \ln t - t)$. Since $t \ln t \rightarrow 0$, the limit is 0 and the integral equals -1 .

15 Drug exposure



By parts, with $u = 8t$, $dv = e^{-0.5t} dt$, $v = -2e^{-0.5t}$:

$$\int 8te^{-0.5t} dt = -16te^{-0.5t} + 16 \int e^{-0.5t} dt = -16te^{-0.5t} - 32e^{-0.5t}.$$

At ∞ the value is 0 and at 0 it is -32 . The total exposure is **32** mg·h/L.

16 A cycle of exponential and cosine



Let $I = \int e^{2x} \cos x dx$ and $J = \int e^{2x} \sin x dx$. Parts gives $I = \frac{1}{2}e^{2x} \cos x + \frac{1}{2}J$ and $J = \frac{1}{2}e^{2x} \sin x - \frac{1}{2}I$.

Substitute: $I = \frac{1}{2}e^{2x} \cos x + \frac{1}{4}e^{2x} \sin x - \frac{1}{4}I$, so $\frac{5}{4}I = e^{2x}(\frac{1}{2} \cos x + \frac{1}{4} \sin x)$.

$$I = \frac{e^{2x}(2 \cos x + \sin x)}{5} + C.$$

17 Tangent substitution with a square



Let $x = 3 \tan \theta$, $dx = 3 \sec^2 \theta d\theta$, $\sqrt{x^2 + 9} = 3 \sec \theta$. Then

$$\int \frac{3 \sec^2 \theta}{9 \tan^2 \theta \cdot 3 \sec \theta} d\theta = \frac{1}{9} \int \frac{\cos \theta}{\sin^2 \theta} d\theta = -\frac{1}{9 \sin \theta} + C.$$

From the triangle, $\sin \theta = \frac{x}{\sqrt{x^2 + 9}}$, so $\int = -\frac{\sqrt{x^2 + 9}}{9x} + C$.

18 Substitution then parts

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Let $u = x^2$, $du = 2x dx$: $\int x^3 e^{x^2} dx = \frac{1}{2} \int u e^u du$. By parts, $\int u e^u du = (u - 1)e^u$.

So $\int x^3 e^{x^2} dx = \frac{1}{2}(x^2 - 1)e^{x^2} + C$.

19 Partial fractions and a limit

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Decompose: $\frac{1}{x(x^2 + 1)} = \frac{1}{x} - \frac{x}{x^2 + 1}$. An antiderivative is

$$\ln x - \frac{1}{2} \ln(x^2 + 1) = \ln \frac{x}{\sqrt{x^2 + 1}}.$$

As $b \rightarrow \infty$, $\frac{b}{\sqrt{b^2 + 1}} \rightarrow 1$, so the value at b tends to $\ln 1 = 0$. At 1 it is $\ln \frac{1}{\sqrt{2}} = -\frac{1}{2} \ln 2$.

The integral converges to $\frac{1}{2} \ln 2 \approx 0.347$.

20 Estimating pi

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$h = 0.25$. Values: $f(0) = 4$, $f(0.25) = 3.7647$, $f(0.5) = 3.2$, $f(0.75) = 2.56$, $f(1) = 2$.

$$T_4 = 0.125 [4 + 2(3.7647 + 3.2 + 2.56) + 2] \approx 3.1312 \text{ (error about 0.0104).}$$

$$S_4 = \frac{0.25}{3} [4 + 4(3.7647) + 2(3.2) + 4(2.56) + 2] \approx 3.14157 \text{ (error about 0.00002).}$$

Simpson's rule is more than 400 times more accurate here.

21 Telescoping improper integral

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$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$, so an antiderivative is $\ln \frac{x}{x+1}$. At ∞ it tends to $\ln 1 = 0$; at 1 it is $\ln \frac{1}{2}$. The integral equals $0 - \ln \frac{1}{2} = \ln 2 \approx 0.693$.

For $x \geq 1$, $x^2 + x + 1 > x(x+1)$, so $0 < \frac{1}{x^2 + x + 1} < \frac{1}{x(x+1)}$. By comparison, the second integral converges (and is less than $\ln 2$).

22 A slow divergence



Let $u = \ln x$, $du = \frac{dx}{x}$: $\int_2^b \frac{dx}{x \ln x} = \left[\ln(\ln x) \right]_2^b = \ln(\ln b) - \ln(\ln 2)$.

As $b \rightarrow \infty$, $\ln b \rightarrow \infty$ and so does $\ln(\ln b)$. The integral diverges, although very slowly: even for $b = e^{1000}$ the value is only about 7.27.



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