

Techniques of Integration: math lesson, College – download the PDF

Some integrals fall to a quick guess; most do not. In this chapter you build a toolbox of six methods (parts, trigonometric integrals, trigonometric substitution, partial fractions, improper integrals, and numerical rules) and, just as important, you learn how to pick the right tool in a few seconds.

1. Why you need a toolbox

The Fundamental Theorem of Calculus says that an integral is easy once you know an antiderivative. The difficulty is finding one. Differentiation follows a short list of rules and always works; integration has no single rule, so we rewrite the integrand until it matches something familiar. Each technique below is a different way of rewriting.

Keep the basic forms ready: $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$, $\int \frac{dx}{x} = \ln|x| + C$, $\int e^{kx} dx = \frac{e^{kx}}{k} + C$, and $\int \cos x dx = \sin x + C$. The ordinary u -substitution (for example $\int xe^{x^2} dx = \frac{1}{2}e^{x^2} + C$) is always the first thing to test.

2. Integration by parts

The product rule $(uv)' = u'v + uv'$ integrates to a formula that trades one integral for another, hopefully simpler, one.

P

Integration by parts

$$\int u \, dv = uv - \int v \, du, \quad \int_a^b u v' \, dx = [uv]_a^b - \int_a^b u' v \, dx.$$

M

Method

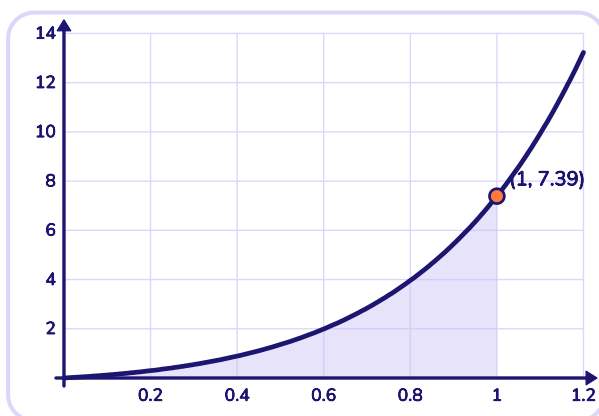
1. Split the integrand into a part u that gets simpler when differentiated and a part dv you can integrate.
2. Write du and v .
3. Apply the formula and finish the new integral.

Good choices for u , in rough order of priority: a logarithm, an inverse trig function, a polynomial, then a trig or exponential factor.

E

Example 1: polynomial times exponential Compute $\int_0^1 x e^{2x} \, dx$. Take $u = x$, $dv = e^{2x} \, dx$, so $du = dx$ and $v = \frac{1}{2} e^{2x}$. Then

$$\int_0^1 x e^{2x} \, dx = \left[\frac{x}{2} e^{2x} \right]_0^1 - \frac{1}{2} \int_0^1 e^{2x} \, dx = \frac{e^2}{2} - \frac{e^2 - 1}{4} = \frac{e^2 + 1}{4} \approx 2.097.$$



E **Example 2: a logarithm alone** For $\int \ln x \, dx$ take $u = \ln x$, $dv = dx$. Then $du = \frac{dx}{x}$, $v = x$ and $\int \ln x \, dx = x \ln x - \int 1 \, dx = x \ln x - x + C$.

E **Example 3: going around in a circle** For $I = \int e^x \cos x \, dx$, parts twice gives $I = e^x \cos x + e^x \sin x - I$. The original integral reappears, so solve for it:

$$I = \frac{e^x(\cos x + \sin x)}{2} + C.$$

3. Trigonometric integrals

Integrals made of powers of sine and cosine are handled by the identities $\sin^2 x + \cos^2 x = 1$, $\sin^2 x = \frac{1 - \cos 2x}{2}$ and $\cos^2 x = \frac{1 + \cos 2x}{2}$.

M **Method for $\int \sin^m x \cos^n x \, dx$**

- If one exponent is odd, peel off one factor of that function, convert the rest with $\sin^2 + \cos^2 = 1$, and substitute for the other function.
- If both exponents are even, lower the powers with the half-angle identities.
- For **tan** and **sec**, use $\tan^2 x = \sec^2 x - 1$ and $\frac{d}{dx} \tan x = \sec^2 x$.

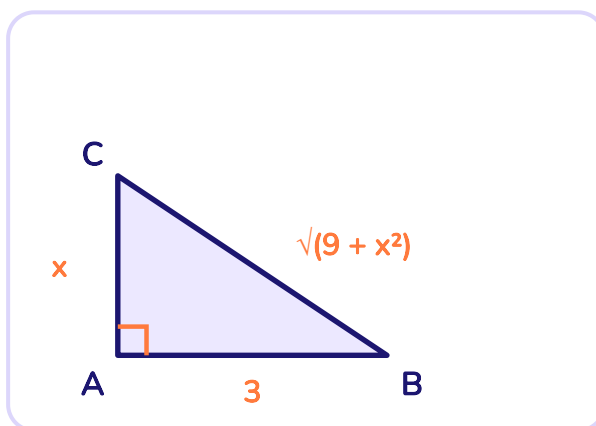
E **Example 4: odd power** $\int \sin^3 x \cos^2 x \, dx = \int (1 - \cos^2 x) \cos^2 x \sin x \, dx$. With $u = \cos x$, $du = -\sin x \, dx$: $-\int (u^2 - u^4) du = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$.

E **Example 5: even power** $\int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$.

4. Trigonometric substitution

When the integrand contains $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$ or $\sqrt{x^2 - a^2}$, a trigonometric identity can remove the root.

Expression in the integrand	Substitution	Identity used
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$1 - \sin^2 \theta = \cos^2 \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$\sec^2 \theta - 1 = \tan^2 \theta$



E **Example 6** Compute $\int \sqrt{9 - x^2} dx$. Let $x = 3 \sin \theta$, $dx = 3 \cos \theta d\theta$, $\sqrt{9 - x^2} = 3 \cos \theta$. Then $\int 9 \cos^2 \theta d\theta = \frac{9}{2}(\theta + \sin \theta \cos \theta) + C$. Returning to x with $\sin \theta = \frac{x}{3}$ and $\cos \theta = \frac{\sqrt{9 - x^2}}{3}$:

$$\int \sqrt{9 - x^2} dx = \frac{9}{2} \arcsin \frac{x}{3} + \frac{x}{2} \sqrt{9 - x^2} + C.$$

E **Example 7** Compute $\int \frac{dx}{(x^2 + 4)^{3/2}}$. Let $x = 2 \tan \theta$, $dx = 2 \sec^2 \theta d\theta$,
 $(x^2 + 4)^{3/2} = 8 \sec^3 \theta$. The integral becomes
 $\int \frac{\cos \theta}{4} d\theta = \frac{\sin \theta}{4} + C = \frac{x}{4\sqrt{x^2 + 4}} + C$, read off a right triangle with legs x and 2.

⚠ **Watch out** Always convert back to x (or change the limits to θ). A triangle sketch is the safest way to do it, and it avoids sign mistakes.

5. Partial fractions

A rational function $\frac{P(x)}{Q(x)}$ with $\deg P < \deg Q$ can be split into simple pieces. If $\deg P \geq \deg Q$, divide first.

M Method

1. Factor $Q(x)$ completely.
2. Each linear factor $(x - r)$ gives $\frac{A}{x - r}$; a repeated factor $(x - r)^2$ gives $\frac{A}{x - r} + \frac{B}{(x - r)^2}$; an irreducible quadratic gives $\frac{Ax + B}{x^2 + px + q}$.
3. Clear denominators and find the constants (plug in roots, or compare coefficients).
4. Integrate each piece.

E **Example 8** $\frac{5x + 7}{(x - 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 3}$. Then
 $5x + 7 = A(x + 3) + B(x - 1)$. At $x = 1$: $12 = 4A$, so $A = 3$. At $x = -3$:
 $-8 = -4B$, so $B = 2$. Therefore
 $\int \frac{5x + 7}{(x - 1)(x + 3)} dx = 3 \ln |x - 1| + 2 \ln |x + 3| + C$.

E Example 9: repeated factor $\frac{3x+1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$ gives $B = 1, C = -2$, $A = 2$. So $\int = 2 \ln |x| - \frac{1}{x} - 2 \ln |x+1| + C$.

6. Improper integrals

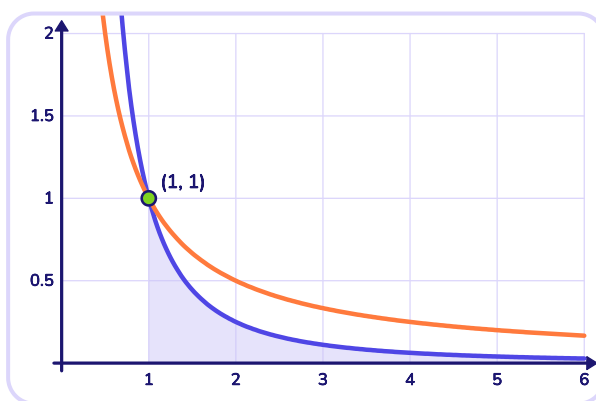
An integral is *improper* if an endpoint is infinite or the integrand blows up on the interval. We replace the trouble spot by a limit.

D Definition

$\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$. If the limit is a finite number, the integral *converges*; otherwise it *diverges*. Similarly, if f blows up at a , use $\lim_{t \rightarrow a^+} \int_t^b f dx$.

P The p-test

$\int_1^\infty \frac{dx}{x^p}$ converges (to $\frac{1}{p-1}$) when $p > 1$ and diverges when $p \leq 1$. Also $\int_0^1 \frac{dx}{x^p}$ converges (to $\frac{1}{1-p}$) when $p < 1$.



E Example 10 $\int_0^\infty e^{-2x} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{2} e^{-2x} \right]_0^b = \frac{1}{2}$. And $\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{t \rightarrow 0^+} [2\sqrt{x}]_t^1 = 2$.

When you cannot find an antiderivative, compare: if $0 \leq f \leq g$ and $\int g$ converges, then $\int f$ converges; if $\int f$ diverges, so does $\int g$.

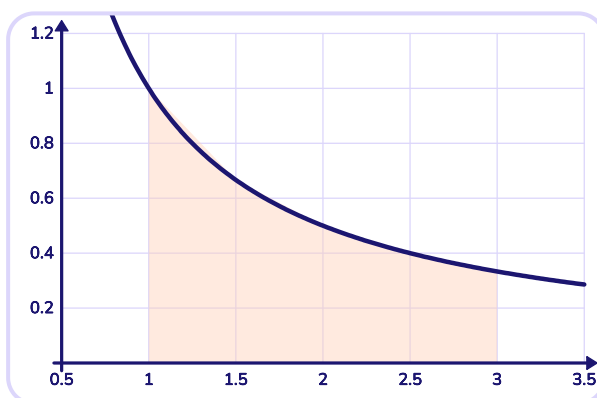
7. Numerical integration

Some antiderivatives (such as that of e^{-x^2}) cannot be written with elementary functions. We then approximate with n subintervals of width $h = \frac{b-a}{n}$ and $x_i = a + ih$.

P Trapezoid and Simpson's rules

$$T_n = \frac{h}{2} [f_0 + 2f_1 + \cdots + 2f_{n-1} + f_n], \quad S_n = \frac{h}{3} [f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots]$$

where $f_i = f(x_i)$ and n must be even for Simpson's rule. The trapezoid error is at most $\frac{K(b-a)^3}{12n^2}$ with $K \geq |f''|$.



E Example 11 Estimate $\ln 3 = \int_1^3 \frac{dx}{x}$ with $n = 4$, $h = 0.5$. The values are $1, \frac{2}{3}, \frac{1}{2}, \frac{2}{5}, \frac{1}{3}$. Then $T_4 = 0.25 \left[1 + 2\left(\frac{2}{3} + \frac{1}{2} + \frac{2}{5}\right) + \frac{1}{3} \right] \approx 1.1167$ and $S_4 = \frac{0.5}{3} \left[1 + \frac{8}{3} + 1 + \frac{8}{5} + \frac{1}{3} \right] = 1.1$. The true value is $\ln 3 \approx 1.0986$, so Simpson's rule is much closer.

8. Choosing a technique

Run through this checklist in order and stop at the first match.

If you see...

Try...

A form that matches a basic antiderivative after a simple change of variable

u -substitution

A product such as xe^x , $x \sin x$, $\ln x$ or $x^n \ln x$

Integration by parts

Powers of $\sin$, $\cos$, $\tan$, $\sec$

Trigonometric identities

$\sqrt{a^2 \pm x^2}$ or $\sqrt{x^2 - a^2}$

Trigonometric substitution

A rational function with a factorable denominator

Partial fractions

An infinite limit or a vertical asymptote

Replace it by a limit (improper integral)

No elementary antiderivative, or only data

Trapezoid or Simpson



Zyro's tip On my home planet we say: simplify first, substitute second, split third. Many "hard" integrals turn easy after a quick algebra step or an obvious u -substitution.



Classic mistakes Forgetting the minus sign in $uv - \int v du$; forgetting to divide first when the numerator's degree is too high; and writing $\int_1^\infty \frac{dx}{x} = 0$ because "1/x goes to 0" (it diverges).

Key takeaways

- Parts: $\int u dv = uv - \int v du$; choose u as the factor that simplifies when differentiated (logs first).
- Odd power of sine or cosine: peel one factor and substitute; even powers: half-angle identities.
- $\sqrt{a^2 - x^2} \rightarrow x = a \sin \theta$; $\sqrt{a^2 + x^2} \rightarrow x = a \tan \theta$; $\sqrt{x^2 - a^2} \rightarrow x = a \sec \theta$.
- Partial fractions: factor, decompose, solve for constants, integrate; divide first if the degree is too big.
- Improper integrals are limits; the p-test: $\int_1^\infty x^{-p} dx$ converges exactly when $p > 1$.
- Trapezoid and Simpson's rules approximate integrals; Simpson's needs an even n and is usually far more accurate.



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