

Integrals and the Fundamental Theorem: math lesson, College – download the PDF

You already know that a derivative measures how fast a quantity changes. An integral does the opposite job: it adds up many tiny changes to find a total. The surprise, and the heart of this chapter, is that these two ideas are inverses of each other. Once you see why, areas, distances, volumes of water and dollars of cost all become one-line calculations.

1. Accumulating a quantity

Suppose a faucet fills a bucket at a steady 2 gallons per minute. After 5 minutes you have added $2 \times 5 = 10$ gallons: rate times time, which is the area of a rectangle. If the rate changes, the region under the rate graph is no longer a rectangle, yet the accumulated amount is still the area under the graph. Integrals are the tool that measures such areas, and they also measure anything that builds up: distance from speed, energy from power, cost from marginal cost.

2. Antiderivatives

D Definition: antiderivative

A function F is an **antiderivative** of f on an interval if $F'(x) = f(x)$ for every x in that interval.

P

Property: the constant of integration

If F is one antiderivative of f , then every antiderivative has the form $F(x) + C$, where C is a constant. The whole family is written $\int f(x) dx = F(x) + C$ and is called the **indefinite integral**.

The constant appears because constants disappear when you differentiate. Here are the antiderivatives you will use most often, each checked by differentiating the right-hand side.

Function $f(x)$	An antiderivative $F(x)$
$x^n (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln x $
$e^{kx} (k \neq 0)$	$\frac{e^{kx}}{k}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$

E Example 1: finding an antiderivative

Find the antiderivative F of $f(x) = 6x^2 - 4x + 5$ with $F(1) = 7$.

Integrate term by term: $F(x) = 2x^3 - 2x^2 + 5x + C$. Then

$F(1) = 2 - 2 + 5 + C = 5 + C$. Setting $5 + C = 7$ gives $C = 2$, so

$F(x) = 2x^3 - 2x^2 + 5x + 2$.

3. Riemann sums

To measure the area under a curve, slice the interval $[a, b]$ into n strips of equal width

$\Delta x = \frac{b-a}{n}$, and replace each strip by a rectangle. With sample points $x_1, \dots, x_n$ the total area of the rectangles is a **Riemann sum**.

D Definition: Riemann sum

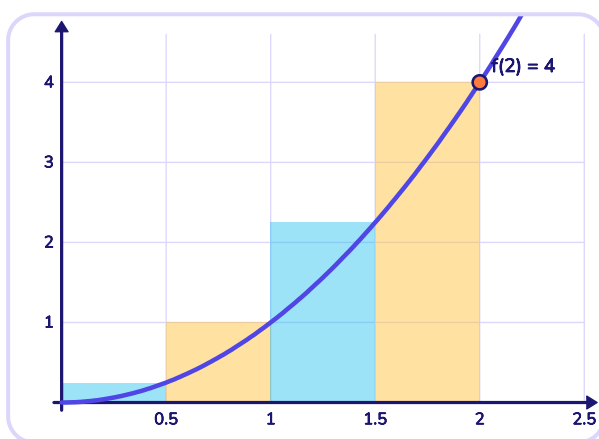
$$S_n = \sum_{i=1}^n f(x_i^*) \Delta x, \quad x_i^* \in [a + (i-1)\Delta x, a + i\Delta x].$$

Choosing the left endpoint, the right endpoint or the midpoint of each strip gives the left, right or midpoint sum.

E Example 2: a right-endpoint sum

Estimate the area under $f(x) = x^2$ on $[0, 2]$ with $n = 4$. Here $\Delta x = 0.5$, and the right endpoints are 0.5, 1, 1.5, 2.

$R_4 = 0.5 (0.25 + 1 + 2.25 + 4) = 0.5 \times 7.5 = 3.75$. The left sum uses 0, 0.5, 1, 1.5 and gives $L_4 = 0.5 (0 + 0.25 + 1 + 2.25) = 1.75$. Because f is increasing, the true area lies between 1.75 and 3.75.



Watch out

Right and left sums over- and underestimate in opposite ways only when the function is monotonic. If f rises and falls, you cannot say which sum is larger without checking the graph.

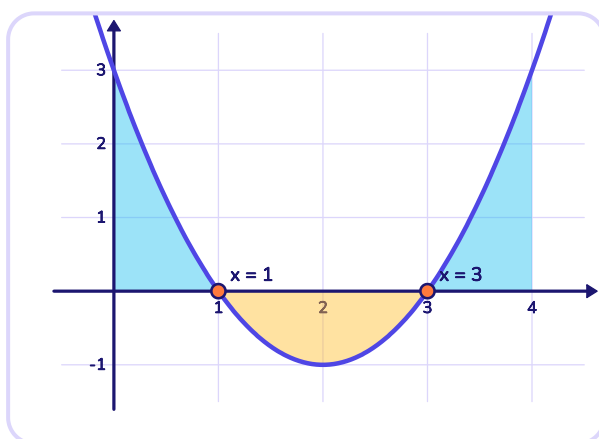
4. The definite integral

D Definition: definite integral

If f is continuous on $[a, b]$, the **definite integral** is the limit of Riemann sums as the strips get thinner:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x.$$

The integral measures **signed area**: regions above the x-axis count as positive, regions below count as negative. For $f(x) = (x - 1)(x - 3)$ on $[0, 4]$ the three pieces are $+\frac{4}{3}$, $-\frac{4}{3}$ and $+\frac{4}{3}$, so the integral equals $\frac{4}{3}$, even though the total unsigned area is $\frac{12}{3} = 4$.



5. Properties of integrals

P

Properties

For continuous f and g and any constant c :

- $\int_a^a f(x) dx = 0$ and $\int_b^a f(x) dx = -\int_a^b f(x) dx$.
- $\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$.
- $\int_a^b c f(x) dx = c \int_a^b f(x) dx$.
- $\int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx$ for any c .
- If $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f dx \geq \int_a^b g dx$.

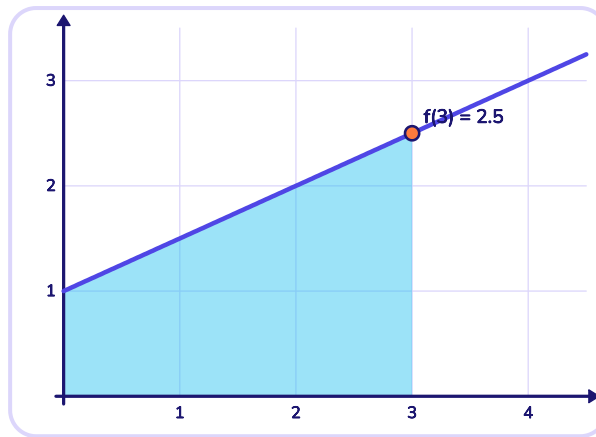


Watch out

There is no product rule of this kind: in general $\int_a^b f(x)g(x) dx \neq \left(\int_a^b f\right) \left(\int_a^b g\right)$.

6. The Fundamental Theorem of Calculus

Let $A(x) = \int_a^x f(t) dt$ be the area accumulated from a up to x . When x grows by a small amount h , the area grows by about a thin rectangle of height $f(x)$ and width h . So $A(x+h) - A(x) \approx f(x)h$, which says $A'(x) = f(x)$.



P Fundamental Theorem of Calculus

Part 1. If f is continuous on $[a, b]$ and $A(x) = \int_a^x f(t) dt$, then $A'(x) = f(x)$. With a variable upper limit $g(x)$, the chain rule gives $\frac{d}{dx} \int_a^{g(x)} f(t) dt = f(g(x)) g'(x)$.

Part 2. If F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a) = \left[F(x) \right]_a^b.$$

M Method: evaluating a definite integral

1. Find an antiderivative F (no constant is needed).
2. Evaluate $F(b)$ and $F(a)$.
3. Subtract: $F(b) - F(a)$, and state units if the problem has them.

E Example 3: using Part 2

Compute $\int_1^3 (4x^3 - 2x + 1) dx$. An antiderivative is $F(x) = x^4 - x^2 + x$. Then $F(3) = 81 - 9 + 3 = 75$ and $F(1) = 1 - 1 + 1 = 1$, so the integral is $75 - 1 = 74$.

! Zyro's tip

On my home planet we say: “integrate, then subtract.” Always write the bracket $[F(x)]_a^b$ before plugging in numbers, and your signs will stay safe!

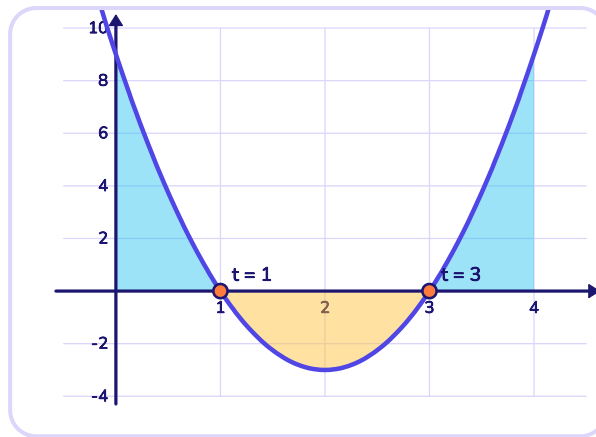
7. Net change

Because $F(b) - F(a) = \int_a^b F'(x) dx$, the integral of a rate of change is the **net change** of the quantity. If $v(t)$ is velocity in meters per second, then $\int_a^b v(t) dt$ is the **displacement**. The **total distance** is $\int_a^b |v(t)| dt$: split the interval where v changes sign and add the absolute values.

E Example 4: displacement and distance

A cart on a track has velocity $v(t) = 3t^2 - 12t + 9$ m/s for $0 \leq t \leq 4$. Since $v(t) = 3(t - 1)(t - 3)$, the cart reverses direction at $t = 1$ and $t = 3$. A position function is $s(t) = t^3 - 6t^2 + 9t$, so $s(0) = 0$, $s(1) = 4$, $s(3) = 0$, $s(4) = 4$.

Displacement: $s(4) - s(0) = 4$ m. Total distance: $4 + |0 - 4| + 4 = 12$ m, which is about 39.4 feet.



8. The substitution rule

The chain rule says $\frac{d}{dx} F(g(x)) = F'(g(x)) g'(x)$. Read backward, it gives the substitution rule.

P Substitution rule

If $u = g(x)$, then $\int f(g(x)) g'(x) dx = \int f(u) du$. For definite integrals, change the limits too:

$$\int_a^b f(g(x)) g'(x) dx = \int_{g(a)}^{g(b)} f(u) du.$$

M Method: substitution

1. Choose u as an inner piece whose derivative also appears in the integrand.
2. Compute $du = g'(x) dx$ and rewrite everything in terms of u .
3. Integrate in u ; for a definite integral, convert the limits to u -values.
4. Return to x (indefinite case only).

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Example 5: definite substitution

Compute $\int_0^2 3x^2(x^3 + 1)^2 dx$. Let $u = x^3 + 1$, so $du = 3x^2 dx$. When $x = 0$, $u = 1$; when $x = 2$, $u = 9$. Then

$$\int_1^9 u^2 du = \left[\frac{u^3}{3} \right]_1^9 = \frac{729 - 1}{3} = \frac{728}{3}.$$

Key takeaways

- An antiderivative of f is a function F with $F' = f$; all of them differ by a constant C .
- A Riemann sum adds up rectangle areas; the definite integral is the limit of these sums and measures signed area.
- Integrals are linear, additive over intervals, and reverse sign when the limits are swapped.
- Fundamental Theorem: $\frac{d}{dx} \int_a^x f(t) dt = f(x)$ and $\int_a^b f(x) dx = F(b) - F(a)$.
- The integral of a rate is the net change; use $|v|$ to get total distance.
- Substitution undoes the chain rule: set $u = g(x)$, replace $g'(x) dx$ by du , and convert the limits.



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