

Polynomial and Rational Functions Review: math lesson, Grade 12 – download the PDF

Polynomials describe roller-coaster tracks, the volume of a box, and the cost of making a product. Rational functions, which are quotients of polynomials, explain why an average cost levels off and why some graphs shoot up toward a wall. In this review you will connect every big idea about these functions: end behavior, zeros, division, the key theorems, asymptotes, and inequalities.

1. Polynomial functions and their parts

D Polynomial function

A polynomial function of degree n has the form

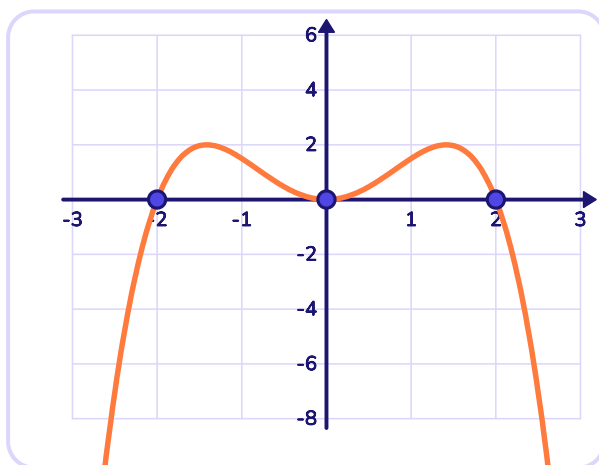
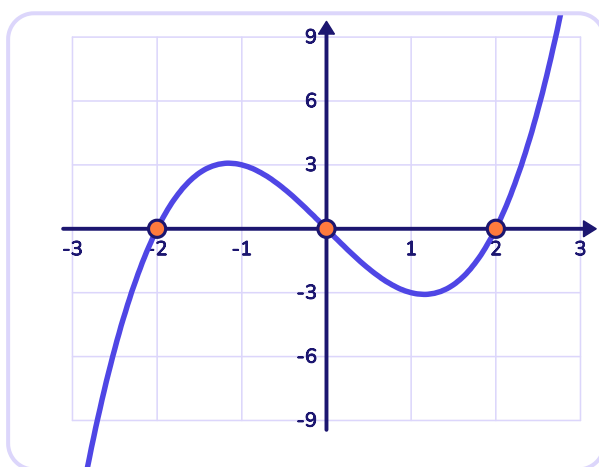
$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $a_n \neq 0$ and the exponents are whole numbers. The number n is the **degree**, a_n is the **leading coefficient**, and a_0 is the **constant term**, which is also the y -intercept $P(0)$.

For example, $P(x) = -2x^5 + 4x^2 - 7$ has degree 5, leading coefficient -2 , and y -intercept -7 . Polynomial graphs are smooth and continuous: no breaks, no corners.

2. End behavior of polynomials

Far from the origin, the term with the highest power wins. So only two facts matter: whether the degree is even or odd, and whether the leading coefficient is positive or negative.

Degree	Leading coefficient	Left end ($x \rightarrow -\infty$)	Right end ($x \rightarrow +\infty$)
even	positive	rises	rises
even	negative	falls	falls
odd	positive	falls	rises
odd	negative	rises	falls



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Example 1: reading end behavior

Describe the end behavior of $f(x) = -2x^5 + 4x^2 - 7$.

The degree is 5 (odd) and the leading coefficient is -2 (negative). So $f(x) \rightarrow +\infty$ as $x \rightarrow -\infty$, and $f(x) \rightarrow -\infty$ as $x \rightarrow +\infty$: the graph rises on the left and falls on the right.

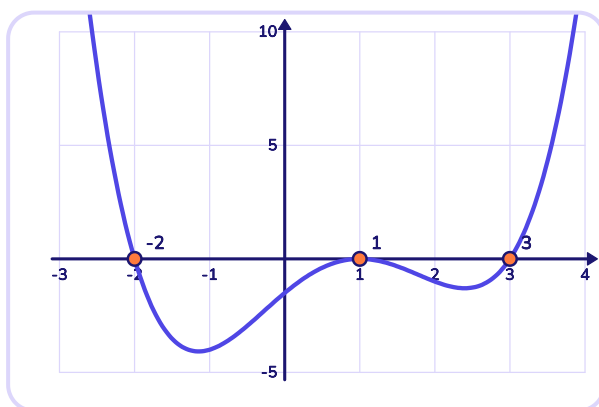
3. Zeros and multiplicity

A **zero** (or root) of P is a number c with $P(c) = 0$. It is an x -intercept of the graph. If $(x - c)^m$ is a factor and $(x - c)^{m+1}$ is not, then c is a zero of **multiplicity** m .

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Multiplicity and the graph

- Odd multiplicity: the graph **crosses** the x -axis at c (and flattens out when $m \geq 3$).
- Even multiplicity: the graph **touches** the axis at c and turns around.
- A polynomial of degree n has at most n real zeros, and at most $n - 1$ turning points.



The graph above is $f(x) = \frac{1}{4}(x+2)(x-1)^2(x-3)$: it crosses at -2 and 3 (multiplicity 1) and touches at 1 (multiplicity 2).

4. Synthetic division

To divide a polynomial by $x - c$, write the coefficients (put a **0** for every missing power), bring down the first one, then repeat: multiply by c , add to the next coefficient. The last number is the remainder; the others are the coefficients of the quotient.

E Example 2: dividing by $x - 3$

Divide $P(x) = 2x^3 - 3x^2 - 11x + 6$ by $x - 3$.

$c = 3$	2	-3	-11	6
Multiply by c		6	9	-6
Add	2	3	-2	0

The quotient is $2x^2 + 3x - 2$ and the remainder is **0**. So

$P(x) = (x - 3)(2x^2 + 3x - 2) = (x - 3)(2x - 1)(x + 2)$, with zeros 3 , $\frac{1}{2}$, and -2 .



Common mistake

To divide by $x + 4$, use $c = -4$, not 4 . And never skip a missing term: $x^3 - 5x + 1$ has coefficients $1, 0, -5, 1$.

5. The Remainder and Factor Theorems

P Remainder Theorem

When a polynomial $P(x)$ is divided by $x - c$, the remainder is $P(c)$.

P Factor Theorem

$x - c$ is a factor of $P(x)$ if and only if $P(c) = 0$.

Together they let you test a possible zero without any long division: just evaluate.

E Example 3: the remainder without dividing

Find the remainder when $Q(x) = x^3 + 4x^2 - 5x - 7$ is divided by $x + 2$.

Here $x + 2 = x - (-2)$, so the remainder is $Q(-2) = -8 + 16 + 10 - 7 = 11$.

6. The Rational Root Theorem

P Rational Root Theorem

If a polynomial with integer coefficients has a rational zero $\frac{p}{q}$ in lowest terms, then p divides the constant term and q divides the leading coefficient.

M**Finding all zeros**

1. List every $\pm \frac{p}{q}$ allowed by the theorem.
2. Test candidates (with the Remainder Theorem or synthetic division) until one gives remainder **0**.
3. Divide it out, then solve the lower-degree quotient.

E**Example 4: all the zeros of a cubic**

Solve $3x^3 - 4x^2 - 5x + 2 = 0$. The constant term **2** has divisors $\pm 1, \pm 2$; the leading coefficient **3** has divisors **1, 3**. The candidates are $\pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3}$. Since $P(2) = 24 - 16 - 10 + 2 = 0$, divide by $x - 2$ to get $3x^2 + 2x - 1 = (3x - 1)(x + 1)$. The zeros are **2**, $\frac{1}{3}$, and **-1**.

!**Zyro's tip**

Try the small integers ± 1 and ± 2 first. They are quick to test and succeed surprisingly often in problems like these.

7. Complex zeros and the Fundamental Theorem of Algebra**P****Fundamental Theorem of Algebra**

Every polynomial of degree $n \geq 1$ has exactly n complex zeros, counting multiplicity. If the coefficients are real, non-real zeros come in conjugate pairs: if $a + bi$ is a zero, so is $a - bi$.

Consequences: a polynomial of odd degree with real coefficients always has at least one real zero, while a polynomial of even degree may have none (think of $x^2 + 1$).

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Example 5: using a conjugate pair

The polynomial $x^3 - 3x^2 + 7x - 5$ has zero $1 + 2i$, so $1 - 2i$ is also a zero. Their quadratic factor is $(x - 1 - 2i)(x - 1 + 2i) = x^2 - 2x + 5$. Dividing gives $x^3 - 3x^2 + 7x - 5 = (x - 1)(x^2 - 2x + 5)$, so the three zeros are 1 , $1 + 2i$, and $1 - 2i$.

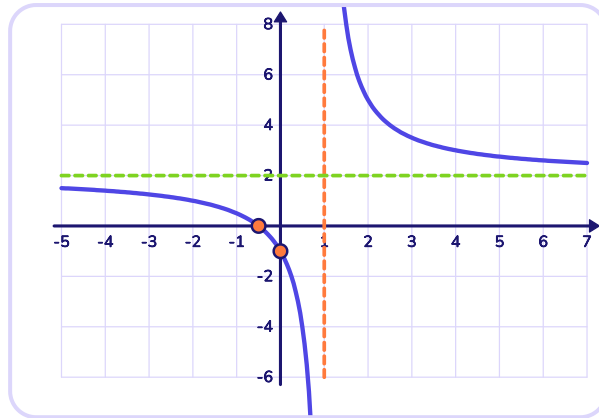
8. Rational functions and asymptotes

A **rational function** is a quotient $f(x) = \frac{N(x)}{D(x)}$ of polynomials. Cancel common factors first, then read the graph.

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Asymptote rules

- **Vertical asymptote** at each zero of D left after cancelling. A cancelled zero leaves a **hole** instead.
- If $\deg N < \deg D$: horizontal asymptote $y = 0$.
- If $\deg N = \deg D$: horizontal asymptote $y =$ (ratio of the leading coefficients).
- If $\deg N = \deg D + 1$: a **slant asymptote**, given by the quotient of the division.



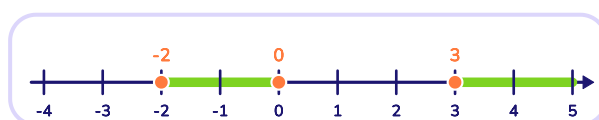
E Example 6: asymptotes and intercepts

For $f(x) = \frac{2x+1}{x-1}$ (graphed above): the denominator is zero at $x = 1$, so the vertical asymptote is $x = 1$. The degrees are equal and $\frac{2}{1} = 2$, so the horizontal asymptote is $y = 2$. The x -intercept solves $2x + 1 = 0$, giving $x = -\frac{1}{2}$; the y -intercept is $f(0) = -1$.

I 9. Solving polynomial and rational inequalities

M Sign-chart method

1. Move everything to one side so the other side is 0.
2. Factor, and find the **critical numbers**: zeros of the numerator and of the denominator.
3. Test one value in each interval, or track the sign of each factor.
4. Keep the intervals with the right sign. Always exclude zeros of the denominator.





Example 7: a polynomial inequality

Solve $x^3 - x^2 - 6x > 0$. Factor: $x(x - 3)(x + 2) > 0$. Critical numbers: $-2, 0, 3$. Testing $x = -3, -1, 1, 4$ gives signs $-, +, -, +$. The solution is $(-2, 0) \cup (3, \infty)$, as shown above.



Example 8: a rational inequality

Solve $\frac{x - 1}{x + 3} \leq 0$. Critical numbers: 1 (included, it makes the fraction 0) and -3 (excluded, division by zero). The signs on $(-\infty, -3)$, $(-3, 1)$, $(1, \infty)$ are $+, -, +$. The solution is $(-3, 1]$.



Common mistake

Do not multiply both sides by an expression whose sign you do not know. Subtract and combine into a single fraction instead.

Key takeaways

- End behavior depends only on the degree parity and the sign of the leading coefficient.
- Odd multiplicity: the graph crosses the axis. Even multiplicity: it touches and turns.
- Synthetic division by $x - c$ gives the quotient and the remainder $P(c)$.
- $x - c$ is a factor of $P(x)$ exactly when $P(c) = 0$.
- Rational zeros are of the form $\frac{p}{q}$ with $p \mid a_0$ and $q \mid a_n$.
- A degree- n polynomial has n complex zeros; for real coefficients, non-real zeros come in conjugate pairs.
- Vertical asymptotes come from the denominator after cancelling; compare degrees for horizontal or slant asymptotes.
- Solve inequalities with critical numbers and a sign chart, never excluding a zero of the denominator.



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