

Function Composition and Inverses: math lesson, Grade 12 – download the PDF

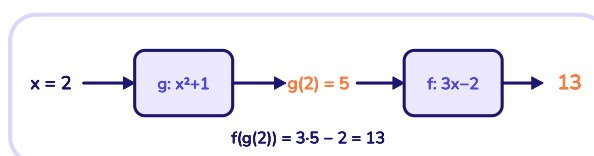
Functions are machines, and in Precalculus you learn to chain them, to run them backward, and to reshape their graphs on purpose. This chapter gives you four powerful tools: **composition**, **inverses**, **transformations**, and **symmetry**. Together they let you build complicated functions from simple pieces and take them apart again.

1. Composition of functions

When the output of one function becomes the input of another, you are **composing** them. Think of two machines placed one after the other on an assembly line.

D Definition: composite function

The composition of f and g is the function $(f \circ g)(x) = f(g(x))$. You apply g first, then f . Read $f \circ g$ as “ f of g ”.



E**Example 1: composing two functions**

Let $f(x) = 3x - 2$ and $g(x) = x^2 + 1$.

$$(f \circ g)(x) = f(x^2 + 1) = 3(x^2 + 1) - 2 = 3x^2 + 1.$$

$$(g \circ f)(x) = g(3x - 2) = (3x - 2)^2 + 1 = 9x^2 - 12x + 5.$$

At $x = 2$: $(f \circ g)(2) = 3(4) + 1 = 13$ and $(g \circ f)(2) = 36 - 24 + 5 = 17$. The two results differ.

A**Order matters**

In general $f \circ g \neq g \circ f$. Always work from the inside out: the function written on the right acts first.

It is just as useful to go the other way and **decompose** a function. For $h(x) = (5x + 4)^3$, take the inner function $g(x) = 5x + 4$ and the outer function $f(x) = x^3$, so that $h = f \circ g$.

2. Domain of a composite function

A composite function can only be evaluated when every step makes sense. That gives two conditions.

P**Domain of $f \circ g$**

A number x belongs to the domain of $f \circ g$ exactly when (1) x is in the domain of g , and (2) $g(x)$ is in the domain of f .

M**Method: find the domain of $f \circ g$**

1. Write down the domain of the inner function g .
2. Find which values of x make $g(x)$ land in the domain of f .
3. Keep only the values that satisfy both conditions. Do this *before* simplifying the formula.

E**Example 2: a domain that simplification would hide**

Let $f(x) = \frac{1}{x-3}$ and $g(x) = \sqrt{x}$. Then $(f \circ g)(x) = \frac{1}{\sqrt{x}-3}$.

Condition 1: $x \geq 0$. Condition 2: $g(x) \neq 3$, that is $\sqrt{x} \neq 3$, so $x \neq 9$.

The domain is $[0, 9) \cup (9, \infty)$.

3. One-to-one functions and the horizontal line test**D****Definition: one-to-one function**

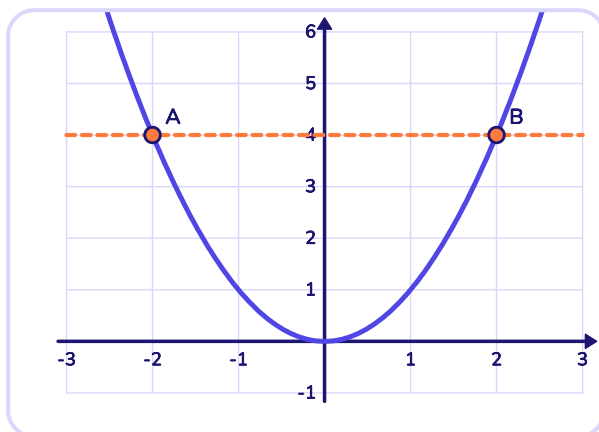
A function f is **one-to-one** if different inputs always give different outputs:

$$f(a) = f(b) \text{ implies } a = b.$$

You can check this on a graph. If some horizontal line meets the graph twice, two different inputs share an output, so the function is not one-to-one.

P Horizontal line test

A function is one-to-one if and only if every horizontal line crosses its graph at most once.



The parabola $y = x^2$ fails the test: the line $y = 4$ meets it at $A(-2, 4)$ and $B(2, 4)$. By contrast $y = x^3$ and every non-constant linear function pass. A function that is strictly increasing, or strictly decreasing, on its domain is one-to-one.

! Zyro's trick

On my planet we repair a failing function by cutting its domain in half. Restrict x^2 to $x \geq 0$ and the horizontal line test is passed!

I 4. Finding an inverse function

If f is one-to-one, it can be “undone”. The function that undoes it is its inverse.

D Definition: inverse function

If f is one-to-one, its inverse f^{-1} is the function that satisfies $f^{-1}(f(x)) = x$ for every x in the domain of f , and $f(f^{-1}(y)) = y$ for every y in the range of f . The domain of f^{-1} is the range of f , and the range of f^{-1} is the domain of f .

⚠ Not a reciprocal

The symbol $f^{-1}(x)$ does *not* mean $\frac{1}{f(x)}$. It is a notation for the inverse function.

M Method: find f^{-1}

1. Check that f is one-to-one (restrict its domain if needed).
2. Write $y = f(x)$.
3. Solve this equation for x in terms of y .
4. Swap the names: write the result as $f^{-1}(x)$.
5. Verify with $f(f^{-1}(x)) = x$ and state the domain.

E Example 3: a rational function

Let $f(x) = \frac{2x+1}{x-3}$ for $x \neq 3$. Set $y = \frac{2x+1}{x-3}$, so $y(x-3) = 2x+1$, then $xy - 3y = 2x+1$ and $x(y-2) = 3y+1$. Hence $x = \frac{3y+1}{y-2}$.

So $f^{-1}(x) = \frac{3x+1}{x-2}$ with $x \neq 2$. Check: $f(5) = \frac{11}{2}$ and $f^{-1}\left(\frac{11}{2}\right) = \frac{35/2}{7/2} = 5$.

E Example 4: restricting the domain first

The function $f(x) = x^2 - 4x + 7$ is not one-to-one, but on $x \geq 2$ it is. Complete the square: $f(x) = (x-2)^2 + 3$, with range $[3, \infty)$.

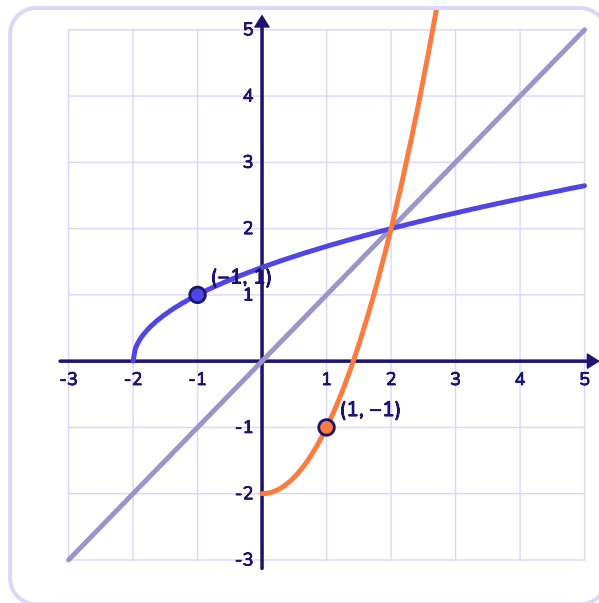
From $y = (x-2)^2 + 3$ with $x \geq 2$: $x = 2 + \sqrt{y-3}$. So $f^{-1}(x) = 2 + \sqrt{x-3}$, with domain $[3, \infty)$ and range $[2, \infty)$.

I 5. Graphs of inverses

If the point (a, b) lies on the graph of f , then $f(a) = b$, so $f^{-1}(b) = a$ and the point (b, a) lies on the graph of f^{-1} .

P Symmetry of inverse graphs

The graph of f^{-1} is the reflection of the graph of f across the line $y = x$.



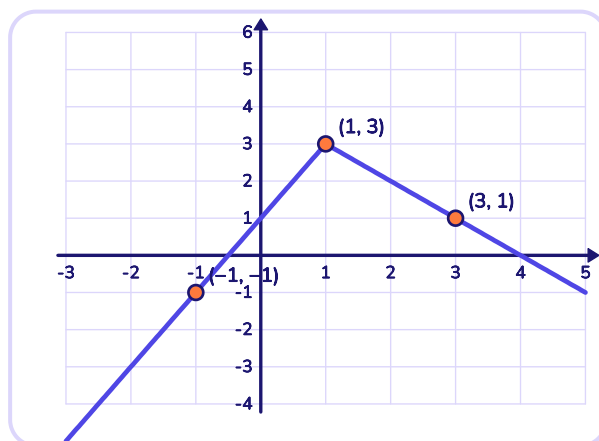
In the figure, the purple curve is $f(x) = \sqrt{x+2}$, whose inverse is $f^{-1}(x) = x^2 - 2$ for $x \geq 0$ (orange curve). The gray line is $y = x$. Notice that $(-1, 1)$ on f matches $(1, -1)$ on f^{-1} . The domain $[-2, \infty)$ and range $[0, \infty)$ of f have swapped roles for f^{-1} .

6. Piecewise and absolute value functions

A **piecewise function** uses different formulas on different parts of its domain. The absolute value function is the most famous example:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Geometrically, $|x - c|$ is the distance between x and c . To solve $|x - c| = d$ with $d > 0$, split into $x - c = d$ or $x - c = -d$.



The graph above is $p(x) = 2x + 1$ for $x < 1$ and $p(x) = 4 - x$ for $x \geq 1$. Both pieces give **3** at $x = 1$, so there is no jump, only a corner.

E Example 5: rewriting a sum of absolute values

Write $h(x) = |x - 2| + |x + 1|$ without absolute values. The critical numbers are -1 and 2 .

- If $x < -1$: $h(x) = -(x - 2) - (x + 1) = 1 - 2x$.
- If $-1 \leq x < 2$: $h(x) = (2 - x) + (x + 1) = 3$.
- If $x \geq 2$: $h(x) = (x - 2) + (x + 1) = 2x - 1$.

Check at $x = 3$: $|1| + |4| = 5 = 2(3) - 1$.

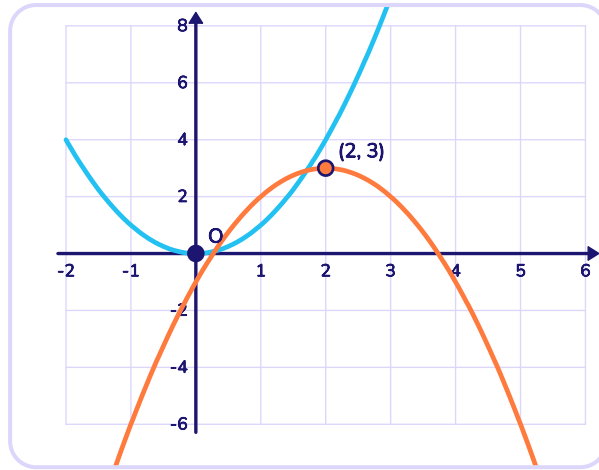
7. Transformations of parent graphs

Every function in a family comes from a **parent function** such as x^2 , x^3 , $\sqrt{x}$, $|x|$ or $\frac{1}{x}$. The function $y = a f(x - h) + k$ is obtained from $y = f(x)$ by simple moves.

P Transformation rules

- h : shift right by h (left if $h < 0$); k : shift up by k (down if $k < 0$).
- a : vertical stretch by $|a|$ if $|a| > 1$, shrink if $|a| < 1$; reflection in the x-axis if $a < 0$.
- $y = f(bx)$: horizontal compression by the factor $\frac{1}{|b|}$ if $|b| > 1$ (stretch if $|b| < 1$); reflection in the y-axis if $b < 0$.

A point (x, y) on the parent graph moves to $(x + h, a y + k)$ on the image of $y = a f(x - h) + k$.



E Example 6: describing a transformation

Start from $y = x^2$ and obtain $g(x) = -(x - 2)^2 + 3$: shift right 2, reflect in the x -axis, shift up 3. The vertex moves from $(0, 0)$ to $(2, 3)$, and the parabola opens downward. The x -intercepts satisfy $(x - 2)^2 = 3$, so $x = 2 \pm \sqrt{3}$.

8. Even and odd functions

D Definition: even and odd

- f is **even** if $f(-x) = f(x)$ for all x in its domain (the graph is symmetric about the y -axis).
- f is **odd** if $f(-x) = -f(x)$ for all x in its domain (the graph is symmetric about the origin).

M**Method: test the symmetry**

1. Compute $f(-x)$ and simplify.
2. Compare with $f(x)$ and with $-f(x)$.
3. If neither matches, the function is neither even nor odd. Remember that a single counterexample is enough to disprove a symmetry.

E**Example 7: three functions**

$f(x) = x^4 - 3x^2$: $f(-x) = x^4 - 3x^2 = f(x)$, so it is even.

$g(x) = x^3 - 5x$: $g(-x) = -x^3 + 5x = -g(x)$, so it is odd.

$h(x) = x^2 + x$: $h(1) = 2$ but $h(-1) = 0$, which is neither **2** nor **-2**, so h is neither.

Useful facts: the product of two odd functions is even; the product of an even and an odd function is odd; and if an odd function is defined at **0**, then $f(0) = 0$.

Key takeaways

- $(f \circ g)(x) = f(g(x))$: the inner function acts first, and usually $f \circ g \neq g \circ f$.
- The domain of $f \circ g$ needs x in the domain of g and $g(x)$ in the domain of f .
- One-to-one means every horizontal line meets the graph at most once; only such functions have inverses.
- To find f^{-1} : solve $y = f(x)$ for x , then swap names; domain and range exchange roles.
- The graph of f^{-1} is the mirror image of the graph of f across $y = x$.
- For piecewise and absolute value functions, split at the critical numbers and treat each case.
- $y = af(x - h) + k$ shifts, stretches and reflects the parent graph;
 $(x, y) \rightarrow (x + h, ay + k)$.
- Even: $f(-x) = f(x)$. Odd: $f(-x) = -f(x)$.



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