

Exponential and Logarithmic Functions: math lesson, Grade 12 – download the PDF

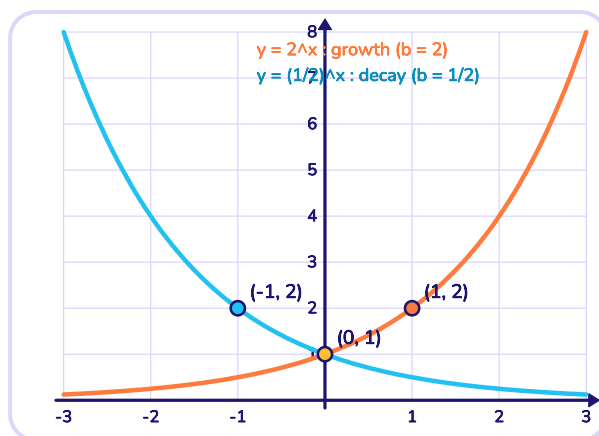
Money in a savings account, bacteria in a dish, a medicine leaving your body, the acidity of a lemon: all of these change by a *percentage* at each step rather than by a fixed amount. Exponential functions model that behavior, and logarithms are the tool that lets you solve for the unknown time or rate. In this chapter you will build both ideas from scratch and use them to solve real problems.

1. Exponential growth and decay

D Exponential function

An exponential function has the form $f(x) = a \cdot b^x$, where $a \neq 0$ is the **initial value** (the value at $x = 0$) and $b > 0$, $b \neq 1$ is the **growth factor**. The variable is in the *exponent*.

If $b > 1$ the function models **growth**; if $0 < b < 1$ it models **decay**. When a quantity changes by a rate r per period, the factor is $b = 1 + r$: a 7% gain gives $b = 1.07$, and a 15% loss gives $b = 0.85$.



Both curves pass through $(0, 1)$, have the x-axis as a horizontal asymptote, and never touch it. Their domain is all real numbers and their range is $y > 0$ (when $a > 0$).

E Example 1: doubling bacteria

A culture starts with 500 bacteria and doubles every 3 hours, so $N(t) = 500 \cdot 2^{t/3}$. After 12 hours: $N(12) = 500 \cdot 2^4 = 500 \cdot 16 = 8,000$ bacteria.

E Example 2: depreciation

A car worth \$24,000 loses 15% of its value each year: $V(t) = 24000 \cdot 0.85^t$. After 3 years: $V(3) = 24000 \cdot 0.614125 = \$14,739$.

2. The natural base e

Among all possible bases, one is special. Compound a 100% yearly rate n times a year and let n grow without bound: $\left(1 + \frac{1}{n}\right)^n$ approaches the irrational number

$$e \approx 2.71828.$$

The function $f(x) = e^x$ is the exponential whose graph has slope 1 at the point $(0, 1)$, which is

why calculus loves it. Quantities that grow or decay *continuously* follow $P(t) = P_0 e^{kt}$: growth if $k > 0$, decay if $k < 0$.

E Example 3: continuous growth

Invest \$2,000 at 4% compounded continuously for 10 years:

$$A = 2000 e^{0.04 \cdot 10} = 2000 e^{0.4} \approx 2000(1.49182) \approx \$2,983.65.$$

I 3. Logarithms as inverses of exponentials

D Logarithm

For $b > 0$, $b \neq 1$ and $x > 0$:

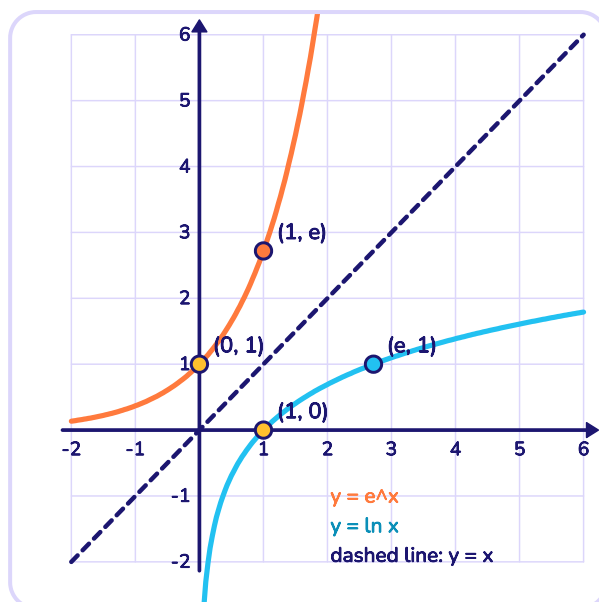
$$\log_b x = y \iff b^y = x.$$

The logarithm answers the question: *to what power must I raise b to get x ?* We write $\log x$ for base 10 and $\ln x$ for base e .

Examples: $\log_2 32 = 5$ because $2^5 = 32$; $\log 0.001 = -3$; $\log_9 3 = \frac{1}{2}$ because $9^{1/2} = 3$.

P Inverse properties

$b^{\log_b x} = x$ for $x > 0$, and $\log_b(b^x) = x$ for every real x . Also $\log_b 1 = 0$ and $\log_b b = 1$.



The graphs of $y = e^x$ and $y = \ln x$ are mirror images across the line $y = x$, because the two functions undo each other. The domain of $\ln x$ is $x > 0$ and its range is all real numbers; the y -axis is a vertical asymptote.



Domain check

You can never take the logarithm of zero or a negative number. For example, $\ln(x - 4)$ is defined only when $x > 4$.

4. Properties of logarithms



Laws of logarithms

For $M, N > 0$ and any real number p :

- Product: $\log_b(MN) = \log_b M + \log_b N$
- Quotient: $\log_b \frac{M}{N} = \log_b M - \log_b N$
- Power: $\log_b(M^p) = p \log_b M$

These laws come straight from the laws of exponents: adding exponents corresponds to multiplying numbers.

E Example 4: expanding and condensing

Expand: $\ln(12) = \ln(2^2 \cdot 3) = 2 \ln 2 + \ln 3 \approx 2(0.6931) + 1.0986 = 2.4849$.

Condense: $3 \log_5 2 + \log_5 25 - \log_5 4 = \log_5 \frac{2^3 \cdot 25}{4} = \log_5 50$.

A Classic mistake

$\log(a + b)$ is *not* $\log a + \log b$, and $\frac{\log a}{\log b}$ is *not* $\log \frac{a}{b}$. The laws apply to products, quotients and powers only.

5. Change of base formula

Calculators only have **log** and **ln**. To evaluate another base, use

$$\log_b x = \frac{\ln x}{\ln b} = \frac{\log x}{\log b}.$$

Proof: if $y = \log_b x$ then $b^y = x$; taking **ln** of both sides gives $y \ln b = \ln x$.

E Example 5

$\log_7 50 = \frac{\ln 50}{\ln 7} \approx \frac{3.9120}{1.9459} \approx 2.0104$. Check: $7^2 = 49$, just below 50, so the answer should be a little above 2.

6. Solving exponential equations

M

Method

1. Isolate the exponential expression.
2. If both sides can be written with the same base, set the exponents equal.
3. Otherwise take **ln** (or **log**) of both sides and bring the exponent down with the power law.
4. Solve for the variable and check with a calculator.

E

Example 6

$$(a) 3^{2x-1} = 81 = 3^4 \Rightarrow 2x - 1 = 4 \Rightarrow x = \frac{5}{2}.$$

$$(b) 5 \cdot 2^x = 100 \Rightarrow 2^x = 20 \Rightarrow x = \frac{\ln 20}{\ln 2} \approx 4.3219.$$

$$(c) e^{3x} = 14 \Rightarrow 3x = \ln 14 \Rightarrow x = \frac{\ln 14}{3} \approx 0.8797.$$

$$(d) 4^x - 6 \cdot 2^x + 8 = 0. \text{ Let } u = 2^x:$$

$$u^2 - 6u + 8 = 0 \Rightarrow (u - 2)(u - 4) = 0 \Rightarrow u = 2 \text{ or } u = 4. \text{ So } x = 1 \text{ or } x = 2.$$

7. Solving logarithmic equations

M

Method

1. Use the laws to condense everything into a single logarithm.
2. Rewrite in exponential form (or equate the arguments of two logs with the same base).
3. Solve, then **check each answer in the original equation**: arguments must be positive.

E

Example 7

(a) $\log_2(x + 3) = 5 \Rightarrow x + 3 = 2^5 = 32 \Rightarrow x = 29$.

(b)

$\log x + \log(x - 3) = 1 \Rightarrow \log[x(x - 3)] = 1 \Rightarrow x^2 - 3x = 10 \Rightarrow (x - 5)(x + 2) = 0$
. Since x must exceed 3, reject $x = -2$. Answer: $x = 5$.

!

Zyro's tip

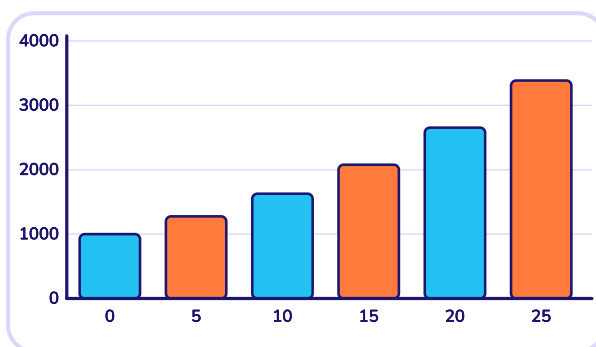
On my home planet we say: "solve first, *then* test." A logarithm equation can invent solutions that your domain rules throw out. Always plug back in!

8. Compound interest and modeling

P Interest formulas

Principal P , annual rate r (as a decimal), n compounding periods per year, t years:

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{and, for continuous compounding,} \quad A = Pe^{rt}.$$



The bars show \$1,000 at 5% compounded annually: each 5-year step multiplies the balance by $1.05^5 \approx 1.276$, so the bars grow faster and faster.

E Example 8: monthly compounding

\$5,000 at 6% compounded monthly for 8 years: $A = 5000(1.005)^{96} \approx \$8,070.71$.

E Example 9: time needed

How long for \$12,000 at 3.5% compounded quarterly to reach \$20,000? Solve

$$12000(1.00875)^{4t} = 20000: (1.00875)^{4t} = \frac{5}{3}, \text{ so } 4t = \frac{\ln(5/3)}{\ln 1.00875} \text{ and } t \approx 14.66 \text{ years.}$$

Doubling time. With continuous growth at rate r , $Pe^{rt} = 2P$ gives $t = \frac{\ln 2}{r}$. **Half-life.** A substance with half-life h follows $N(t) = N_0\left(\frac{1}{2}\right)^{t/h}$.

Key takeaways

- $f(x) = ab^x$ models growth when $b > 1$ and decay when $0 < b < 1$; $b = 1 + r$.
- $e \approx 2.71828$; continuous change follows P_0e^{kt} .
- $\log_b x = y \iff b^y = x$; $\ln$ and e^x are inverses, and $\log_b x$ needs $x > 0$.
- Product, quotient and power laws turn multiplication into addition.
- Change of base: $\log_b x = \frac{\ln x}{\ln b}$.
- To solve exponentials, match bases or take logs; to solve logarithms, condense, exponentiate and *check the domain*.
- $A = P(1 + r/n)^{nt}$ and $A = Pe^{rt}$ are the two interest models.



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