

# Eigenvalues and Orthogonality: practice solutions, College – download the PDF

Written solutions to the chapter problems. Check each step, then correct yourself.

## 1 Is it an eigenvector?



(a)  $A\mathbf{v} = (1 + 2, 3 + 0) = (3, 3) = 3\mathbf{v}$ . Yes, it is an eigenvector with eigenvalue **3**.

(b)  $A\mathbf{w} = (1 + 4, 3) = (5, 3)$ . This is not a multiple of  $(1, 2)$ , because  $5/1 \neq 3/2$ . So  $\mathbf{w}$  is not an eigenvector.

## 2 Dot product and length



(a)  $\mathbf{u} \cdot \mathbf{w} = 2 \cdot 4 + (-1) \cdot 5 + 3 \cdot (-1) = 8 - 5 - 3 = 0$ . The vectors are orthogonal.

(b)  $\|\mathbf{u}\| = \sqrt{4 + 1 + 9} = \sqrt{14} \approx 3.74$ .

## 3 A first characteristic polynomial



The trace is **9** and the determinant is  $20 - 2 = 18$ . So

$\det(A - \lambda I) = \lambda^2 - 9\lambda + 18 = (\lambda - 3)(\lambda - 6)$ . The eigenvalues are **3** and **6**. Check:  $3 + 6 = 9$  and  $3 \cdot 6 = 18$ .

#### 4 Triangular matrix



$T$  is upper triangular, so its eigenvalues are the diagonal entries: **2**, **-3**, **4**. Check with the trace:  $2 + (-3) + 4 = 3$ , which equals the sum of the diagonal entries  $2 - 3 + 4 = 3$ .

#### 5 Angle between two vectors



$\mathbf{u} \cdot \mathbf{w} = 0 + 0 + 1 = 1$  and  $\|\mathbf{u}\| = \|\mathbf{w}\| = \sqrt{2}$ . So  $\cos \theta = \frac{1}{\sqrt{2} \cdot \sqrt{2}} = \frac{1}{2}$ , and  $\theta = 60^\circ$ .

#### 6 A quick projection



$\mathbf{b} \cdot \mathbf{a} = 12 + 2 = 14$  and  $\mathbf{a} \cdot \mathbf{a} = 5$ . So  $\text{proj}_{\mathbf{a}} \mathbf{b} = \frac{14}{5}(2, 1) = (5.6, 2.8)$ .

#### 7 True or false: powers and shifts



$A^2\mathbf{v} = A(4\mathbf{v}) = 4A\mathbf{v} = 16\mathbf{v}$ . Then  $A^3\mathbf{v} = 4 \cdot 16\mathbf{v} = 64\mathbf{v}$ . And  $(A + 3I)\mathbf{v} = 4\mathbf{v} + 3\mathbf{v} = 7\mathbf{v}$ .

True: if  $A\mathbf{v} = 0\mathbf{v} = \mathbf{0}$  with  $\mathbf{v} \neq \mathbf{0}$ , then  $A$  sends a nonzero vector to zero, so it cannot be invertible (equivalently,  $\det A = 0$ ).

#### 8 Finding eigenvectors



The trace is **7** and the determinant is  $12 - 2 = 10$ . So  $\lambda^2 - 7\lambda + 10 = (\lambda - 5)(\lambda - 2)$ .

For  $\lambda = 5$ :  $A - 5I = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$  gives  $y = x$ , so  $(1, 1)$ .

For  $\lambda = 2$ :  $A - 2I = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$  gives  $y = -2x$ , so  $(1, -2)$ .

Check:  $A(1, -2) = (4 - 2, 2 - 6) = (2, -4) = 2(1, -2)$ .

## 9 Diagonalize and take a power



(a) Trace 7, determinant 6:  $\lambda^2 - 7\lambda + 6 = (\lambda - 6)(\lambda - 1)$ . For  $\lambda = 6$ :

$$A - 6I = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \text{ gives } (2, 1). \text{ For } \lambda = 1: A - I = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \text{ gives } (1, -2).$$

$$(b) P = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}, D = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}.$$

(c)  $A^3 = PD^3P^{-1}$ , so its eigenvalues are  $6^3 = 216$  and  $1^3 = 1$ . Directly,

$$A^2 = \begin{pmatrix} 29 & 14 \\ 14 & 8 \end{pmatrix} \text{ and } A^3 = A^2A = \begin{pmatrix} 173 & 86 \\ 86 & 44 \end{pmatrix}. \text{ Check: trace } 217 = 216 + 1 \text{ and determinant } 7612 - 7396 = 216.$$

## 10 A 3 by 3 characteristic polynomial



Expanding along the third row,

$$\det(M - \lambda I) = (4 - \lambda)[(1 - \lambda)^2 - 4] = (4 - \lambda)(\lambda - 3)(\lambda + 1). \text{ The eigenvalues are } 4, 3, -1.$$

$$\lambda = 3: (1, 1, 0). \lambda = -1: (1, -1, 0). \lambda = 4: (0, 0, 1).$$

$$\text{Check with the trace: } 4 + 3 - 1 = 6 = 1 + 1 + 4.$$

## 11 A rotation has no real eigenvalues



$\det(R - \lambda I) = \lambda^2 + 1$ . This equals 0 only for  $\lambda = \pm i$ , which are not real. So there is no real eigenvalue.

Geometrically, rotating by  $90^\circ$  turns every nonzero vector off its line, so no direction is sent to a multiple of itself.

## 12 Gram-Schmidt in the plane



$\mathbf{u}_1 = (1, 2)$ .  $\mathbf{v}_2 \cdot \mathbf{u}_1 = 6$ ,  $\mathbf{u}_1 \cdot \mathbf{u}_1 = 5$ , so

$$\mathbf{u}_2 = (4, 1) - \frac{6}{5}(1, 2) = \left(\frac{14}{5}, -\frac{7}{5}\right) = \frac{7}{5}(2, -1).$$

Check:  $(1, 2) \cdot (2, -1) = 0$ . Both  $(1, 2)$  and  $(2, -1)$  have length  $\sqrt{5}$ , so an orthonormal basis is  $\frac{1}{\sqrt{5}}(1, 2)$  and  $\frac{1}{\sqrt{5}}(2, -1)$ .

## 13 Distance from a point to a line



$\mathbf{b} = (1, 7)$ ,  $\mathbf{a} = (1, 2)$ :  $\mathbf{b} \cdot \mathbf{a} = 15$ ,  $\mathbf{a} \cdot \mathbf{a} = 5$ . The closest point is  $3\mathbf{a} = (3, 6)$ .

The error is  $(1, 7) - (3, 6) = (-2, 1)$ ; check  $(-2, 1) \cdot (1, 2) = 0$ . The distance is  $\sqrt{4 + 1} = \sqrt{5} \approx 2.24$ , that is about 224 feet (about 68 meters).

## 14 A three-point fit



$$A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}, \mathbf{b} = (1, 2, 6). \text{ Then } A^T A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } A^T \mathbf{b} = \begin{pmatrix} 9 \\ 5 \end{pmatrix}.$$

So  $3a = 9$  and  $2b = 5$ :  $a = 3$ ,  $b = 2.5$ . The line is  $y = 3 + 2.5x$ .

## 15 Diagonalizable or not?



(a) The characteristic polynomial is  $(\lambda - 3)^2$ , so 3 is the only eigenvalue.

$A - 3I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  forces  $y = 0$ , so all eigenvectors are multiples of  $(1, 0)$ . There is only one independent eigenvector, so  $A$  is not diagonalizable.

(b)  $B$  is already diagonal ( $B = 3I$ ), so it is diagonalizable: every nonzero vector is an eigenvector. Repeated eigenvalues alone do not decide the question; the number of independent eigenvectors does.

## 16 Finding a missing entry

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5 is an eigenvalue exactly when  $\det(A - 5I) = 0$ :  $(1 - 5)(4 - 5) - 2k = 4 - 2k = 0$ , so  $k = 2$ .

The trace is 5, so the other eigenvalue is  $5 - 5 = 0$ . Check:  $\det A = 4 - 4 = 0$  and  $5 \cdot 0 = 0$ .

## 17 Orthogonal diagonalization of a symmetric matrix

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(a)  $S(1, 1, 1) = (4, 4, 4) = 4(1, 1, 1)$ , eigenvalue 4.

(b)  $S(1, -1, 0) = (2 - 1, 1 - 2, 0) = (1, -1, 0)$ .

$S(1, 1, -2) = (2 + 1 - 2, 1 + 2 - 2, 1 + 1 - 4) = (1, 1, -2)$ . Both have eigenvalue 1.

Dot products:  $(1, 1, 1) \cdot (1, -1, 0) = 0$ ,  $(1, 1, 1) \cdot (1, 1, -2) = 0$ ,

$(1, -1, 0) \cdot (1, 1, -2) = 0$ .

(c) The eigenvalues of  $S$  are 4, 1, 1, so those of  $S^5$  are  $4^5, 1, 1$  and the trace is  $1024 + 1 + 1 = 1026$ .

## 18 Gram-Schmidt in three dimensions

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$\mathbf{u}_1 = (1, 1, 0)$ . Since  $\mathbf{v}_2 \cdot \mathbf{u}_1 = 1$  and  $\mathbf{u}_1 \cdot \mathbf{u}_1 = 2$ ,

$\mathbf{u}_2 = (1, 0, 1) - \frac{1}{2}(1, 1, 0) = (\frac{1}{2}, -\frac{1}{2}, 1)$ .

For  $\mathbf{v}_3$ :  $\mathbf{v}_3 \cdot \mathbf{u}_1 = 1$  and  $\mathbf{v}_3 \cdot \mathbf{u}_2 = \frac{1}{2}$  with  $\mathbf{u}_2 \cdot \mathbf{u}_2 = \frac{3}{2}$ . So

$$\mathbf{u}_3 = (0, 1, 1) - \frac{1}{2}(1, 1, 0) - \frac{1}{3}(\frac{1}{2}, -\frac{1}{2}, 1) = (-\frac{2}{3}, \frac{2}{3}, \frac{2}{3}).$$

Check:  $\mathbf{u}_3 \cdot \mathbf{u}_1 = 0$  and  $\mathbf{u}_3 \cdot \mathbf{u}_2 = -\frac{1}{3} - \frac{1}{3} + \frac{2}{3} = 0$ .

### 19 Least squares for plant growth

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(a)  $n = 4$ ,  $\sum x = 6$ ,  $\sum y = 9$ ,  $\sum x^2 = 14$ ,  $\sum xy = 0 + 2 + 4 + 12 = 18$ . The normal equations  $4a + 6b = 9$ ,  $6a + 14b = 18$  give  $b = \frac{4 \cdot 18 - 6 \cdot 9}{4 \cdot 14 - 36} = \frac{18}{20} = 0.9$  and  $a = \frac{9 - 5.4}{4} = 0.9$ . So  $y = 0.9 + 0.9x$ .

(b) The predicted values are 0.9, 1.8, 2.7, 3.6, so the residuals are 0.1, 0.2, -0.7, 0.4 and the sum of squares is  $0.01 + 0.04 + 0.49 + 0.16 = 0.70$ .

(c) At  $x = 5$ :  $0.9 + 4.5 = 5.4$  inches (about 13.7 cm).

### 20 Eigenvectors of symmetric matrices

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Since  $A$  is symmetric,  $(Au) \cdot w = u \cdot (Aw)$ . The left side is  $\lambda(u \cdot w)$  and the right side is  $\mu(u \cdot w)$ . Hence  $(\lambda - \mu)(u \cdot w) = 0$ . Because  $\lambda \neq \mu$ , we must have  $u \cdot w = 0$ .

### 21 A projection matrix

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(a)  $\mathbf{a} \cdot \mathbf{a} = 9$ , so  $P = \frac{1}{9} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix}$ . Since  $\mathbf{b} \cdot \mathbf{a} = 3 + 0 + 6 = 9$ ,  $P\mathbf{b} = \frac{9}{9}\mathbf{a} = (1, 2, 2)$ .

(b)  $P\mathbf{a} = \mathbf{a}$ , so 1 is an eigenvalue. Every vector orthogonal to  $\mathbf{a}$  is sent to  $\mathbf{0}$ , a plane of eigenvectors for 0. The eigenvalues are 1, 0, 0; the trace  $\frac{1+4+4}{9} = 1$  agrees.

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## Two apps, long-term behavior

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(a) Trace 1.7, determinant  $0.72 - 0.02 = 0.70$ :  $\lambda^2 - 1.7\lambda + 0.7 = (\lambda - 1)(\lambda - 0.7)$ . For  $\lambda = 1$ :  $(M - I)\mathbf{v} = \mathbf{0}$  gives  $\mathbf{v}_1 = (2, 1)$ . For  $\lambda = 0.7$ :  $M - 0.7I = \begin{pmatrix} 0.2 & 0.2 \\ 0.1 & 0.1 \end{pmatrix}$  gives  $\mathbf{v}_2 = (1, -1)$ .

(b) Solving  $c_1(2, 1) + c_2(1, -1) = (150, 150)$ :  $2c_1 + c_2 = 150$  and  $c_1 - c_2 = 150$ , so  $c_1 = 100$ ,  $c_2 = -50$ . Then  $\mathbf{x}_n = 100(2, 1) - 50(0.7)^n(1, -1)$ . Check  $n = 1$ :  $(200 - 35, 100 + 35) = (165, 135)$ , the same as  $M\mathbf{x}_0$ .

(c) Since  $(0.7)^n \rightarrow 0$ ,  $\mathbf{x}_n \rightarrow (200, 100)$ : about 200,000 users on A and 100,000 on B, whatever the split was at the start, with total 300,000 unchanged.



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