

Eigenvalues and Orthogonality: math practice, College – download the PDF

22 problems graded by difficulty (★ easy, ★★ medium, ★★★ hard). Try first, without looking at the solutions!

1 Is it an eigenvector?

★ ★ ★

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$. For each vector, decide whether it is an eigenvector of A . If so, give its eigenvalue.

a. $\mathbf{v} = (1, 1)$

b. $\mathbf{w} = (1, 2)$

2 Dot product and length

★ ★ ★

Let $\mathbf{u} = (2, -1, 3)$ and $\mathbf{w} = (4, 5, -1)$.

a. Compute $\mathbf{u} \cdot \mathbf{w}$. Are the vectors orthogonal?

b. Compute $\|\mathbf{u}\|$.

3 A first characteristic polynomial

★ ★ ★

Find the eigenvalues of $A = \begin{pmatrix} 5 & 2 \\ 1 & 4 \end{pmatrix}$.

4 Triangular matrix



Without expanding a determinant, find the eigenvalues of $T = \begin{pmatrix} 2 & 7 & 1 \\ 0 & -3 & 5 \\ 0 & 0 & 4 \end{pmatrix}$, and verify them with the trace.

5 Angle between two vectors



Find the angle between $\mathbf{u} = (1, 0, 1)$ and $\mathbf{w} = (0, 1, 1)$.

6 A quick projection



Project $\mathbf{b} = (6, 2)$ onto the line spanned by $\mathbf{a} = (2, 1)$.

7 True or false: powers and shifts



Suppose $A\mathbf{v} = 4\mathbf{v}$ for a nonzero vector $\mathbf{v}$. Compute $A^2\mathbf{v}$, $A^3\mathbf{v}$ and $(A + 3I)\mathbf{v}$ in terms of $\mathbf{v}$. Then decide: is it true that if 0 is an eigenvalue of A , then A is not invertible?

8 Finding eigenvectors



Find the eigenvalues and one eigenvector for each, for $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$.

9 Diagonalize and take a power



Let $A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$.

- Find its eigenvalues and eigenvectors.
- Write $A = PDP^{-1}$.
- Use the eigenvalues to explain why A^3 has eigenvalues **216** and **1**, and compute A^3 .

10 A 3 by 3 characteristic polynomial



Let $M = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$. Find the characteristic polynomial, the eigenvalues, and an eigenvector for each eigenvalue.

11 A rotation has no real eigenvalues



Let $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, a quarter-turn counterclockwise in the plane. Show that R has no real eigenvalue, and explain this geometrically.

12 Gram-Schmidt in the plane



Apply Gram-Schmidt to $\mathbf{v}_1 = (1, 2)$ and $\mathbf{v}_2 = (4, 1)$, then give an orthonormal basis.

13 Distance from a point to a line



A drone at position $(1, 7)$ wants to know how far it is from the straight path through the origin in the direction $(1, 2)$ (units: hundreds of feet). Find the closest point on the path and the distance.

14 A three-point fit

★★★

A coffee shop records the number of iced drinks sold (in dozens) at temperatures $-1, 0, 1$ (in tens of degrees from a reference value): $(-1, 1), (0, 2), (1, 6)$. Use the normal equations to find the least squares line $y = a + bx$.

15 Diagonalizable or not?

★★★

Decide whether each matrix is diagonalizable and justify your answer.

a. $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

b. $B = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

16 Finding a missing entry

★★★

The matrix $A = \begin{pmatrix} 1 & k \\ 2 & 4 \end{pmatrix}$ has 5 as an eigenvalue. Find k and the other eigenvalue.

17 Orthogonal diagonalization of a symmetric matrix

★★★

Let $S = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

- Show that $(1, 1, 1)$ is an eigenvector and find its eigenvalue.
- Show that $(1, -1, 0)$ and $(1, 1, -2)$ are eigenvectors with eigenvalue 1 , and that the three vectors are pairwise orthogonal.
- Find the trace of S^5 .

18 Gram-Schmidt in three dimensions

★★★

Apply Gram-Schmidt to $\mathbf{v}_1 = (1, 1, 0)$, $\mathbf{v}_2 = (1, 0, 1)$, $\mathbf{v}_3 = (0, 1, 1)$.

19 Least squares for plant growth

★★★

A botanist measures a seedling's height (in inches) at weeks 0, 1, 2, 3: 1, 2, 2, 4.

- Find the least squares line $y = a + bx$.
- Compute the sum of squared residuals.
- Predict the height at week 5.

20 Eigenvectors of symmetric matrices

★★★

Let A be a symmetric matrix, with $A\mathbf{u} = \lambda\mathbf{u}$, $A\mathbf{w} = \mu\mathbf{w}$ and $\lambda \neq \mu$. Prove that $\mathbf{u} \cdot \mathbf{w} = 0$.

21 A projection matrix

★★★

Let $\mathbf{a} = (1, 2, 2)$ and $P = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a} \cdot \mathbf{a}}$.

- Write P and compute $P\mathbf{b}$ for $\mathbf{b} = (3, 0, 3)$.
- Find the eigenvalues of P .

22

Two apps, long-term behavior

★★★

Each month, **10%** of the users of app A switch to app B, and **20%** of the users of app B switch to app A. Let $\mathbf{x}_n = (a_n, b_n)$ be the number of users (in thousands) after n months, so $\mathbf{x}_{n+1} = M\mathbf{x}_n$ with $M = \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix}$. Initially $\mathbf{x}_0 = (150, 150)$.

- Find the eigenvalues and eigenvectors of M .
- Write $\mathbf{x}_0$ in terms of the eigenvectors and give a formula for $\mathbf{x}_n$.
- What happens as n grows?

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