

Logic, Proofs and Discrete Math: practice solutions, College – download the PDF

Written solutions to the chapter problems. Check each step, then correct yourself.

1

A truth table



We fill the rows in order.

p	q	$\neg q$	$p \wedge \neg q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

The statement is true only when p is true and q is false. (This is also the negation of $p \Rightarrow q$.)

2 Negating statements



(a) “There is a dog in the park that is not on a leash” (the negation of $\forall$ is $\exists$ with a negated property).

(b) By De Morgan: “The pizza is not cold *or* the salad is not warm.”

3 Converse and contrapositive



Converse: “If an integer is divisible by 3, then it is divisible by 6.” Contrapositive: “If an integer is not divisible by 3, then it is not divisible by 6.”

The original is true, so its contrapositive is true. The converse is false: 9 is divisible by 3 but not by 6.

4 Set operations



$A \cap B = \{2, 4, 6\}$; $A \cup B = \{1, 2, 3, 4, 5, 6, 8, 10\}$; $A \setminus B = \{1, 3, 5\}$.

Check: $|A \cup B| = 6 + 5 - 3 = 8$, which matches the list of 8 elements.

5 Outfits



The three choices are independent, so we multiply: $5 \times 4 \times 3 = 60$.

She can make 60 different outfits.

6 Committee or officers



(a) Order does not matter: $C(7, 2) = \frac{7 \cdot 6}{2} = 21$.

(b) Order matters (the roles differ): $P(7, 2) = 7 \cdot 6 = 42$.

7 First terms of a recurrence



$$a_2 = 2 \cdot 3 + 1 = 7, a_3 = 2 \cdot 7 + 1 = 15, a_4 = 2 \cdot 15 + 1 = 31, a_5 = 2 \cdot 31 + 1 = 63.$$

The terms are 3, 7, 15, 31, 63, and $a_5 = 63$.

8 Sum of two odd integers



Let the integers be $2a + 1$ and $2b + 1$ with a, b integers. Their sum is

$$(2a + 1) + (2b + 1) = 2a + 2b + 2 = 2(a + b + 1).$$

Since $a + b + 1$ is an integer, the sum is a multiple of 2, so it is even. ■

9 Quantifier order



(a) True: given x , choose $y = -x$.

(b) False: for any fixed y , take $x = 1 - y$; then $x + y = 1 \neq 0$.

Negation of (b): $\forall y \exists x (x + y \neq 0)$, which is exactly what we just showed.

10 Sum of odd numbers



Base: for $n = 1$ the left side is 1 and the right side is $1^2 = 1$.

Step: assume $1 + 3 + \dots + (2n - 1) = n^2$. Add the next odd number

$$2(n + 1) - 1 = 2n + 1: n^2 + 2n + 1 = (n + 1)^2.$$

So the formula holds for $n + 1$, and by induction for all $n \geq 1$. ■

11 Injective or surjective?



f is injective: if $2a + 3 = 2b + 3$ then $a = b$. **f is not surjective:** every output $2x + 3$ is odd, so 4 is never reached.

g is not injective: $g(2) = g(-2) = 4$. **g is not surjective:** no integer squared equals -1 .

12 Handshake and Euler



Degrees: vertex 1 has neighbors 2, 4, 3 so degree 3; vertex 2 has degree 2; vertex 3 has neighbors 2, 4, 1 so degree 3; vertex 4 has neighbors 3, 1, 5 so degree 3; vertex 5 has degree 1.

Sum: $3 + 2 + 3 + 3 + 1 = 12 = 2 \cdot 6$, and there are 6 edges. The handshake theorem holds.

The odd vertices are 1, 3, 4 and 5: four of them. An Euler path needs 0 or 2 odd vertices, so none exists.

13 Arranging letters



LEVEL has 5 letters with L twice, E twice and V once. Dividing out the repeats:

$$\frac{5!}{2!2!} = \frac{120}{4} = 30.$$

There are 30 distinct arrangements.

14 Two simple recurrences



(a) Arithmetic: $a_n = 5 + 4n$, so $a_{10} = 5 + 40 = 45$ members.

(b) Geometric: $b_n = 3 \cdot 2^n$, so $b_8 = 3 \cdot 256 = 768$ bacteria.

15 Irrationality of the square root of 3



Suppose $\sqrt{3} = \frac{a}{b}$ with a, b positive integers and the fraction in lowest terms. Then $a^2 = 3b^2$, so $3 \mid a^2$, hence $3 \mid a$, say $a = 3k$.

Then $9k^2 = 3b^2$, so $b^2 = 3k^2$, hence $3 \mid b$. Now 3 divides both a and b , contradicting lowest terms.

So $\sqrt{3}$ is irrational. ■

16 An inequality by induction

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Base: $n = 5$: $2^5 = 32 > 25 = 5^2$.

Step: assume $2^n > n^2$ with $n \geq 5$. Then $2^{n+1} = 2 \cdot 2^n > 2n^2$. Now $2n^2 \geq (n+1)^2$ is equivalent to $n^2 \geq 2n + 1$, true for $n \geq 3$ (since $n^2 \geq 3n \geq 2n + 1$).

So $2^{n+1} > (n+1)^2$, and the claim holds for all $n \geq 5$. ■

17 Three overlapping clubs

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Inclusion–exclusion: $|C \cup M \cup S| = 60 + 45 + 40 - 25 - 20 - 15 + 10 = 95$.

So $100 - 95 = 5$ students take none of the three subjects.

18 Binary strings without two 1s in a row

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(a) $s_1 = 2$ (strings 0, 1) and $s_2 = 3$ (00, 01, 10).

(b) For $n \geq 3$, look at the first symbol. If it is 0, the rest is any valid string of length $n - 1$: s_{n-1} ways. If it is 1, the next symbol must be 0 and the rest is any valid string of length $n - 2$: s_{n-2} ways.

(c) $s_3 = 5$, $s_4 = 8$, $s_5 = 13$, $s_6 = 21$, $s_7 = 34$, $s_8 = 55$.

19 Round robin and odd degrees

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(a) This is the number of edges of K_{12} : $C(12, 2) = \frac{12 \cdot 11}{2} = 66$ games.

(b) If all 7 degrees were 3, the sum would be 21. By the handshake theorem the sum must be even (twice the number of edges), but 21 is odd. Contradiction, so no such graph exists.

20 Composition and inverse



(a) Solve $y = 3x - 5$ for x : $x = \frac{y+5}{3}$, so $f^{-1}(x) = \frac{x+5}{3}$.

(b) $(g \circ f)(x) = (3x - 5)^2 + 1 = 9x^2 - 30x + 26$.

$(f \circ g)(x) = 3(x^2 + 1) - 5 = 3x^2 - 2$.

(c) $(g \circ f)(2) = g(1) = 2$ and $(f \circ g)(2) = f(5) = 10$. The two compositions differ, so order matters.

21 A second-order recurrence



Try $a_n = r^n$: $r^2 = 5r - 6$, so $r^2 - 5r + 6 = 0$ and $r = 2$ or $r = 3$. Hence

$a_n = A \cdot 2^n + B \cdot 3^n$.

From $a_0 = 1$: $A + B = 1$. From $a_1 = 4$: $2A + 3B = 4$. Subtracting twice the first equation gives $B = 2$, so $A = -1$.

Closed form: $a_n = 2 \cdot 3^n - 2^n$. Check: $a_2 = 18 - 4 = 14 = 5 \cdot 4 - 6 \cdot 1$;

$a_3 = 54 - 8 = 46 = 5 \cdot 14 - 6 \cdot 4$.



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