

Logic, Proofs and Discrete Math: math lesson, College – download the PDF

Discrete mathematics studies things you can count one by one: statements, integers, sets, networks and sequences. It is the language behind computer science, cryptography and the proofs you will meet in every upper-level math course. In this chapter you will learn to reason precisely, to write short proofs, and to count and model discrete structures.

1. Propositional logic

A **proposition** is a statement that is either true (T) or false (F), never both. “17 is prime” is a proposition; “Close the door” is not. We build new propositions with connectives: **not** ($\neg p$), **and** ($p \wedge q$), **or** ($p \vee q$, inclusive), and the **implication** $p \Rightarrow q$ (“if p then q ”).

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \Rightarrow q$
T	T	F	T	T	T
T	F	F	F	T	F
F	T	T	F	T	T
F	F	T	F	F	T

D Implication and related statements

$p \Rightarrow q$ is false only when p is true and q is false. For $p \Rightarrow q$:

- the **converse** is $q \Rightarrow p$;
- the **contrapositive** is $\neg q \Rightarrow \neg p$, which is always *equivalent* to the original.

P De Morgan's laws

$\neg(p \wedge q) \equiv \neg p \vee \neg q$ and $\neg(p \vee q) \equiv \neg p \wedge \neg q$. Also $\neg(p \Rightarrow q) \equiv p \wedge \neg q$.

E Example 1: negating a statement

“It is sunny and the temperature is above 70°F.” Its negation is “It is not sunny *or* the temperature is at most 70°F.” The converse of “If it rains, the game is canceled” is “If the game is canceled, it rained,” which is not equivalent; the contrapositive “If the game is not canceled, it did not rain” is.

I 2. Quantifiers

A statement about a variable needs quantifiers. The symbol $\forall$ means “for all” and $\exists$ means “there exists.” For example, $\forall x \in \mathbb{R}, x^2 \geq 0$ is true, while $\exists x \in \mathbb{R}, x^2 = -1$ is false.

P**Negating quantifiers**

$\neg(\forall x, P(x)) \equiv \exists x, \neg P(x)$ and $\neg(\exists x, P(x)) \equiv \forall x, \neg P(x)$. To disprove a “for all” claim, one counterexample is enough.

A**Order matters**

Over the integers, $\forall x \exists y (x + y = 0)$ is true (take $y = -x$), but $\exists y \forall x (x + y = 0)$ is false, because one single y cannot work for every x .

3. Direct proof and contradiction

A **direct proof** starts from the hypotheses and reaches the conclusion by valid steps. A proof **by contrapositive** proves $\neg q \Rightarrow \neg p$ instead. A proof **by contradiction** assumes the statement is false and derives something impossible.

E**Example 2: direct proof**

Claim: the product of two odd integers is odd. Write the integers as $2a + 1$ and $2b + 1$. Then $(2a + 1)(2b + 1) = 4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1$, which has the form $2k + 1$. So the product is odd.

E**Example 3: contradiction**

Claim: $\sqrt{2}$ is irrational. Suppose $\sqrt{2} = \frac{a}{b}$ with a, b integers and the fraction in lowest terms. Then $a^2 = 2b^2$, so a^2 is even, so a is even: $a = 2k$. Then $4k^2 = 2b^2$, so $b^2 = 2k^2$ and b is even too. Both are even, which contradicts “lowest terms.” Hence $\sqrt{2}$ is irrational.

4. Mathematical induction

Induction proves a statement $P(n)$ for every integer $n \geq n_0$.

M**Method: induction**

1. **Base case:** check $P(n_0)$.
2. **Inductive step:** assume $P(n)$ for some $n \geq n_0$ (the *inductive hypothesis*) and prove $P(n + 1)$.
3. **Conclude** that $P(n)$ holds for all $n \geq n_0$.

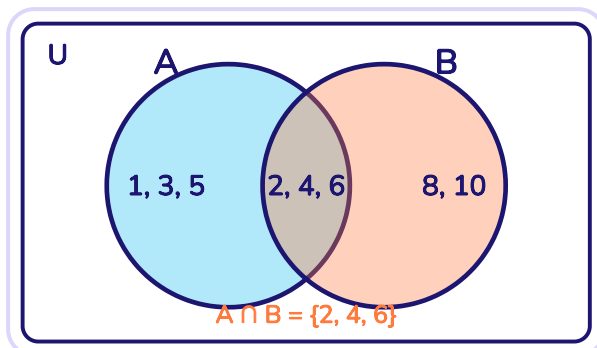
Think of dominoes: the base case knocks over the first one, and the inductive step guarantees each domino knocks over the next.

E**Example 4: divisibility by induction**

Claim: $3 \mid 4^n - 1$ for all $n \geq 1$. Base: $4^1 - 1 = 3$. Step: if $4^n - 1 = 3m$, then $4^{n+1} - 1 = 4(4^n - 1) + 3 = 12m + 3 = 3(4m + 1)$, a multiple of 3. So the claim holds for every $n \geq 1$.

5. Sets and functions

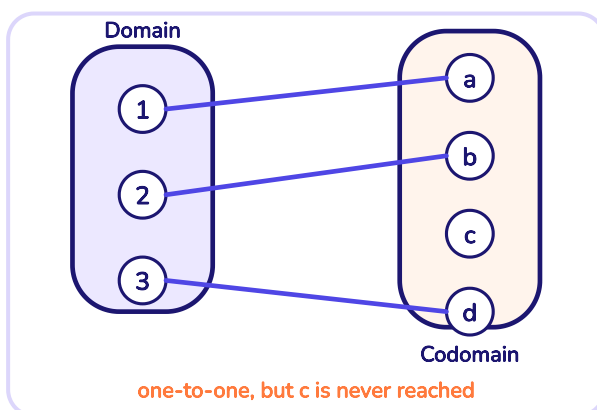
A **set** is a collection of distinct objects. We use union $A \cup B$, intersection $A \cap B$, difference $A \setminus B$ and size $|A|$. For finite sets, **inclusion–exclusion** gives $|A \cup B| = |A| + |B| - |A \cap B|$.



In the diagram, $|A| = 6$, $|B| = 5$, $|A \cap B| = 3$, so $|A \cup B| = 6 + 5 - 3 = 8$.

D Functions

A function $f : X \rightarrow Y$ assigns exactly one output in Y to each input in X . It is **injective** (one-to-one) if $f(a) = f(b) \Rightarrow a = b$; **surjective** (onto) if every element of Y is an output; **bijective** if both. A bijection has an inverse function f^{-1} .



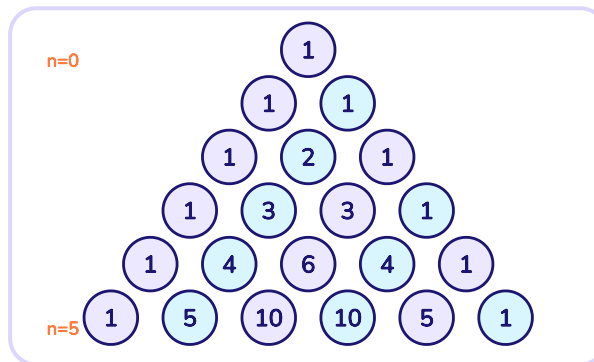
A set with n elements has 2^n subsets, since each element is either in or out.

6. Counting and combinations

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Counting rules

- **Multiplication principle:** independent choices multiply.
- **Permutations:** ordered selections of k from n : $P(n, k) = \frac{n!}{(n-k)!}$.
- **Combinations:** unordered selections: $C(n, k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.



Each number in Pascal's triangle is the sum of the two above it, because

$C(n, k) = C(n-1, k-1) + C(n-1, k)$. When some objects repeat, divide by the factorials of the repeat counts.

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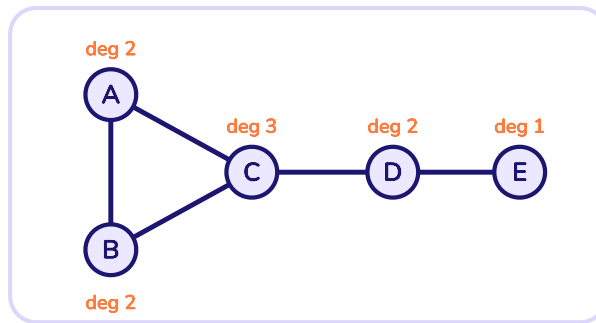
Example 5: committees and passwords

A club of 10 students picks a 3-person committee: $C(10, 3) = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$. If instead it picks a president, a secretary and a treasurer, order matters:

$P(10, 3) = 10 \cdot 9 \cdot 8 = 720$. The letters of "SEEDS" can be arranged in $\frac{5!}{2!2!} = 30$ ways.

7. Graphs basics

A **graph** has **vertices** (dots) joined by **edges** (lines). The **degree** of a vertex is the number of edges at it. A **path** moves along edges; a graph is **connected** if a path joins any two vertices.



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Handshake theorem

The sum of all degrees equals twice the number of edges. So the number of odd-degree vertices is always even. In the graph above, $2 + 2 + 3 + 2 + 1 = 10 = 2 \cdot 5$.

An **Euler path** uses every edge exactly once. A connected graph has one exactly when it has 0 or 2 vertices of odd degree. Here C and E are the only odd vertices, so an Euler path exists and must start at one and end at the other. The complete graph K_n (every pair joined) has $\frac{n(n-1)}{2}$ edges.

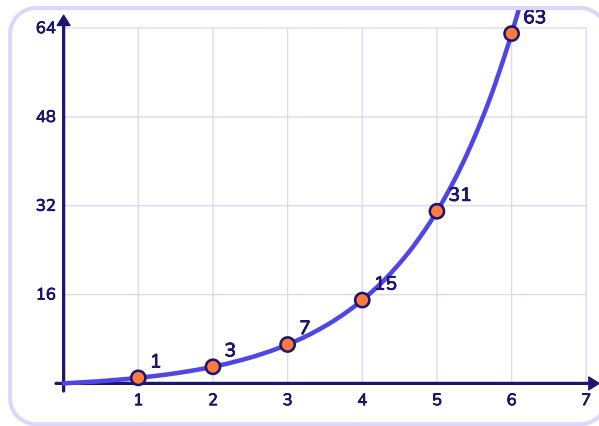
8. Recurrence relations

A **recurrence relation** defines each term from earlier ones, together with starting values. Solving it means finding a **closed form** that gives the n th term directly.

E

Example 6: the Tower of Hanoi count

Moving n disks takes $h_n = 2h_{n-1} + 1$ moves, with $h_1 = 1$. The terms are 1, 3, 7, 15, 31, 63. They match $h_n = 2^n - 1$. Check by induction: $2(2^{n-1} - 1) + 1 = 2^n - 1$.



M

Method: linear recurrences

1. Arithmetic type $a_n = a_{n-1} + d$: $a_n = a_0 + nd$.
2. Geometric type $a_n = r a_{n-1}$: $a_n = a_0 r^n$.
3. For $a_n = \alpha a_{n-1} + \beta a_{n-2}$, try $a_n = r^n$, solve $r^2 = \alpha r + \beta$, and fit the two starting values.

!

Zyro's tip

On my home planet we test every closed form on the first three terms before celebrating. It takes ten seconds and catches almost every slip!

Key takeaways

- $p \Rightarrow q$ is false only when p is true and q is false; it is equivalent to its contrapositive, not to its converse.
- Negating swaps $\forall$ and $\exists$ and negates the inside; one counterexample kills a “for all.”
- Direct proof, contrapositive and contradiction are your three basic proof tools.
- Induction needs a base case and an inductive step.
- $|A \cup B| = |A| + |B| - |A \cap B|$; $P(n, k) = \frac{n!}{(n-k)!}$; $C(n, k) = \frac{n!}{k!(n-k)!}$.
- In any graph, the sum of the degrees is twice the number of edges.
- Test every closed form of a recurrence on the first terms.



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