

Differentiation Rules: math lesson, College – download the PDF

In the last chapter you computed derivatives from the limit definition. That works, but it is slow and easy to get wrong. In this chapter you will build a toolbox of rules that lets you differentiate almost any formula you meet in calculus in a few lines. Each rule comes with a reason, a method, and the classic mistake to avoid.

1. The basic rules we already know

Everything in this chapter sits on top of a few simple facts. Let c be a constant and let f and g be differentiable functions.

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Basic rules

- Constant: $\frac{d}{dx}(c) = 0$.
- Power: $\frac{d}{dx}(x^n) = nx^{n-1}$ for any real number n .
- Constant multiple: $(cf)' = cf'$.
- Sum and difference: $(f \pm g)' = f' \pm g'$.

These rules are linear: they handle sums and constant multiples term by term. They say nothing about products, quotients, or functions placed inside other functions. Those cases need the new rules below.

2. The product rule

Suppose a rectangle has width $f(t)$ and height $g(t)$, and both grow with time. The area fg grows because the width grows *and* because the height grows. That is exactly what the product rule says.

P Product rule

If f and g are differentiable, then

$$(fg)' = f'g + fg'.$$

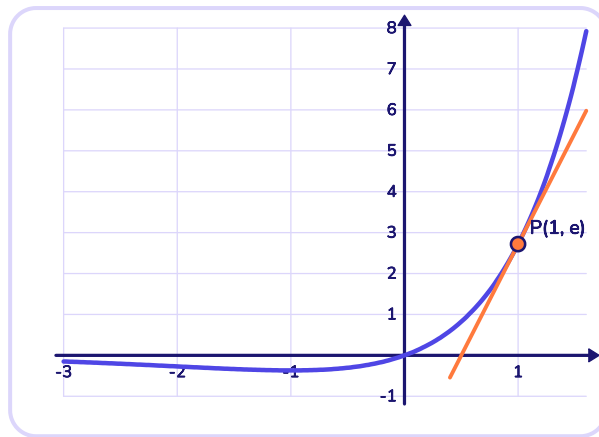
A Common mistake

The derivative of a product is *not* the product of the derivatives. Test it with $x \cdot x = x^2$: the true derivative is $2x$, but $(1)(1) = 1$ is wrong.

E Example 1

Differentiate $h(x) = (x^2 + 3x)e^x$. Take $f = x^2 + 3x$ and $g = e^x$, so $f' = 2x + 3$ and $g' = e^x$.

$$h'(x) = (2x + 3)e^x + (x^2 + 3x)e^x = (x^2 + 5x + 3)e^x.$$



The figure shows $f(x) = xe^x$. Here $f'(x) = 1 \cdot e^x + xe^x = (1+x)e^x$, so the tangent at $P(1, e)$ has slope $f'(1) = 2e \approx 5.44$.

3. The quotient rule

A quotient $\frac{f}{g}$ is a product $f \cdot g^{-1}$, and working it out gives a compact formula.

P Quotient rule

If f and g are differentiable and $g(x) \neq 0$, then

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

! Memory trick

Say it aloud: “low d-high minus high d-low, over low squared.” The order of the two terms on top matters because of the minus sign.

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Example 2

Differentiate $q(x) = \frac{x^2}{x+1}$. Here $f = x^2$, $g = x + 1$.

$$q'(x) = \frac{2x(x+1) - x^2 \cdot 1}{(x+1)^2} = \frac{x^2 + 2x}{(x+1)^2}.$$

4. The chain rule

Many functions are built by plugging one function into another, like $\sqrt{1+x^3}$ or $\sin(5x)$. We call $f(g(x))$ a **composite function**, with g the inner function and f the outer function.

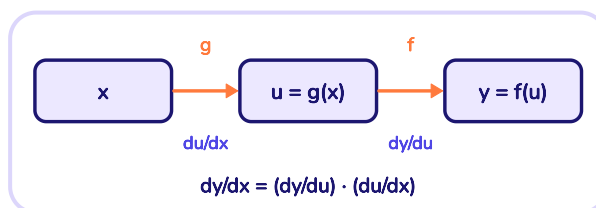
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Chain rule

If g is differentiable at x and f is differentiable at $g(x)$, then

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, with $y = f(u)$ and $u = g(x)$: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.



Think of gears: if u changes 3 times as fast as x and y changes 2 times as fast as u , then y changes 6 times as fast as x . A special case you will use constantly is the general power rule:

$$\frac{d}{dx} (u^n) = n u^{n-1} u'.$$

M**Method: using the chain rule**

1. Identify the outer function and the inner function.
2. Differentiate the outer function, leaving the inner function untouched inside it.
3. Multiply by the derivative of the inner function.
4. Repeat from the outside in if there are more than two layers.

E**Example 3**

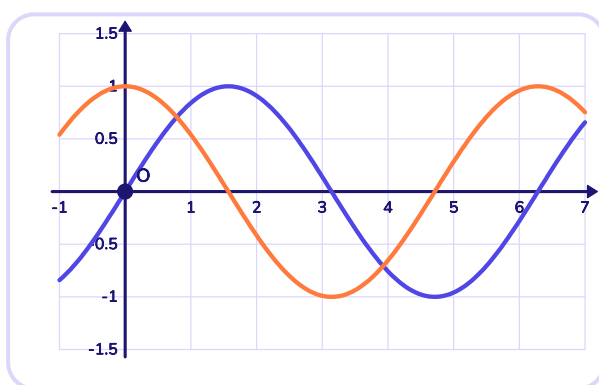
Differentiate $r(x) = \sqrt{1+x^3}$. The outer function is $\sqrt{u}$, the inner function is $u = 1+x^3$.

$$r'(x) = \frac{1}{2\sqrt{1+x^3}} \cdot 3x^2 = \frac{3x^2}{2\sqrt{1+x^3}}.$$

5. Derivatives of trigonometric functions

Throughout calculus, angles are measured in **radians**. These formulas are false in degrees.

Function	Derivative
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$



Look at the graphs: the violet curve **$\sin x$** has a horizontal tangent exactly where the orange curve **$\cos x$** crosses zero, and it climbs fastest at $x = 0$, where **$\cos 0 = 1$** .



Zyro's tip

On my home planet we say: every function that starts with “co” (cosine, cotangent, cosecant) has a derivative with a minus sign. It is a quick way to catch sign errors.

E Example 4

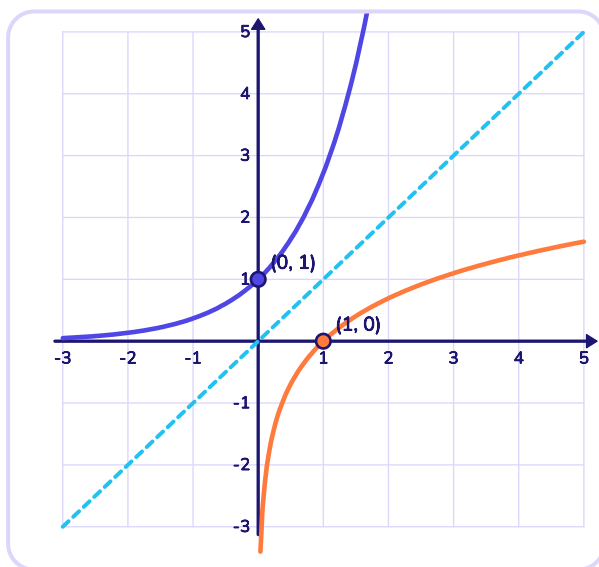
Differentiate $p(x) = x^2 \cos(3x)$. Use the product rule, and the chain rule for $\cos(3x)$.

$$p'(x) = 2x \cos(3x) + x^2 \cdot (-3 \sin(3x)) = 2x \cos(3x) - 3x^2 \sin(3x).$$

6. Derivatives of exponentials and logarithms

P Exponential and logarithm rules

- $\frac{d}{dx}e^x = e^x$ and $\frac{d}{dx}a^x = a^x \ln a$ for $a > 0$.
- $\frac{d}{dx}\ln x = \frac{1}{x}$ and $\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$, for $x > 0$.
- With the chain rule: $\frac{d}{dx}e^u = u'e^u$ and $\frac{d}{dx}\ln u = \frac{u'}{u}$.



The graphs of e^x and $\ln x$ are mirror images across the dashed line $y = x$. At $(0, 1)$ the exponential has slope 1; at $(1, 0)$ the logarithm has slope $\frac{1}{1} = 1$. The slopes match because the tangent lines are also mirror images.

E**Example 5**

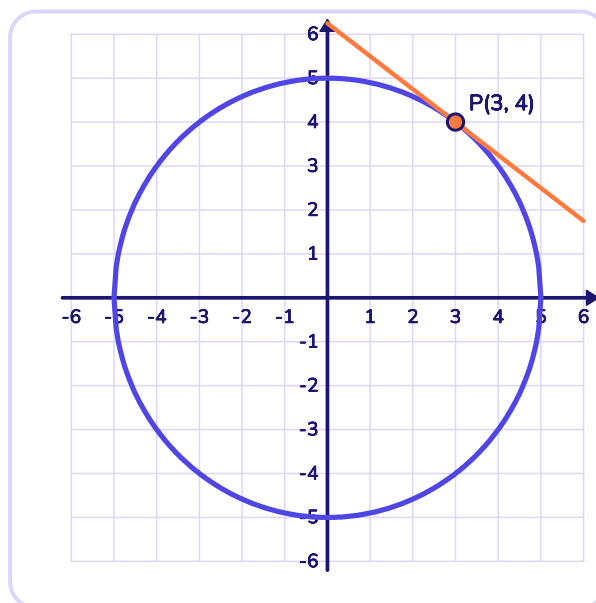
$$(a) \frac{d}{dx} \ln(x^2 + 1) = \frac{2x}{x^2 + 1}. \quad (b) \frac{d}{dx} 5^{3x} = 5^{3x} \cdot \ln 5 \cdot 3 = 3 \ln 5 \cdot 5^{3x}.$$

7. Implicit differentiation

Some curves, like a circle, are not the graph of a single function $y = f(x)$. We can still find slopes by treating y as an unknown function of x and differentiating both sides of the equation.

M**Method: implicit differentiation**

1. Differentiate both sides with respect to x . Every time you differentiate a term with y , the chain rule adds a factor $\frac{dy}{dx}$.
2. Gather all terms containing $\frac{dy}{dx}$ on one side.
3. Factor out $\frac{dy}{dx}$ and divide to solve for it.



E Example 6

Find the slope of the circle $x^2 + y^2 = 25$ at $P(3, 4)$. Differentiating gives

$2x + 2y \frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{x}{y}$. At $(3, 4)$ the slope is $-\frac{3}{4}$, which is the orange tangent in the figure.

E Example 7

The point $(1, 2)$ lies on $y^3 + xy = 10$ since $8 + 2 = 10$. Differentiating:

$3y^2 y' + y + x y' = 0$, so $y' = -\frac{y}{3y^2 + x}$. At $(1, 2)$: $y' = -\frac{2}{13}$.

8. Logarithmic differentiation

Variable exponents such as x^x , or long products and quotients, are painful with the earlier rules. Taking the natural logarithm first turns powers into multiples and products into sums.

M Method: logarithmic differentiation

1. Write $y = \dots$ and take **ln** of both sides, simplifying with log laws.
2. Differentiate both sides implicitly: the left side becomes $\frac{y'}{y}$.
3. Multiply by y and replace y by its original formula.

E**Example 8**

Differentiate $y = x^{x^2}$ for $x > 0$. Then $\ln y = x^2 \ln x$, and differentiating gives

$$\frac{y'}{y} = 2x \ln x + x. \text{ Therefore } y' = x^{x^2} (2x \ln x + x).$$

Common mistake

Do not use the power rule nx^{n-1} when the exponent contains x , and do not use the exponential rule $a^x \ln a$ when the base contains x . If both contain x , use logarithmic differentiation.

9. Higher-order derivatives

Since f' is itself a function, we can differentiate it again to get the **second derivative** f'' , and so on.

D**Notation**

$f''(x) = \frac{d^2y}{dx^2}$, and $f^{(n)}(x) = \frac{d^ny}{dx^n}$ is the n -th derivative. If $s(t)$ is a position, then $s'(t)$ is velocity and $s''(t)$ is acceleration. Graphically, $f'' > 0$ means the curve bends upward and $f'' < 0$ means it bends downward.

E**Example 9**

For $f(x) = \sin(2x)$: $f'(x) = 2 \cos(2x)$ and $f''(x) = -4 \sin(2x)$. For $g(x) = e^{3x}$, each derivative multiplies by 3, so $g^{(n)}(x) = 3^n e^{3x}$.

Key takeaways

- Product rule: $(fg)' = f'g + fg'$. Quotient rule: $(f/g)' = \frac{f'g - fg'}{g^2}$.
- Chain rule: differentiate the outside, keep the inside, multiply by the derivative of the inside.
- Trig derivatives (radians only): $(\sin)' = \cos$, $(\cos)' = -\sin$, $(\tan)' = \sec^2$.
- $(e^x)' = e^x$, $(a^x)' = a^x \ln a$, $(\ln x)' = 1/x$, $(\ln u)' = u'/u$.
- Implicit differentiation: every y -term gets a factor dy/dx ; then solve for it.
- Logarithmic differentiation handles variable exponents and long products: take $\ln$, differentiate, multiply by y .
- Second derivative = acceleration or concavity; $f^{(n)}$ means differentiate n times.



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