

The Derivative and Its Definition: math lesson, College – download the PDF

A speedometer does not tell you your average speed over a trip. It tells you how fast you are going *right now*. Calculus makes that idea precise: the **derivative** measures the instantaneous rate of change of a quantity, and it also gives the slope of a curve at a single point. In this chapter you will build the derivative from a limit, use it to write tangent lines, and learn the first rules that make computing derivatives fast.

1. Average rate of change and the difference quotient

Suppose a function f describes some quantity, such as the height of a drone or the cost of producing items. Between $x = a$ and $x = a + h$, the quantity changes from $f(a)$ to $f(a + h)$.

D Definition: difference quotient

For $h \neq 0$, the **difference quotient** of f at a is

$$\frac{f(a + h) - f(a)}{h}.$$

It is the **average rate of change** of f on the interval between a and $a + h$.

Geometrically, it is the slope of the **secant line** through the points $(a, f(a))$ and $(a + h, f(a + h))$.



E Example 1: a secant slope

Let $f(x) = x^2$ and $a = 1$. Then

$$\frac{f(1+h) - f(1)}{h} = \frac{(1+h)^2 - 1}{h} = \frac{2h + h^2}{h} = 2 + h.$$

For $h = 2$, point B is $(3, 9)$ and the secant slope is 4 , as in the figure. Watch what happens when h shrinks:

h	Point B	Slope of secant AB = $2 + h$
1	(2, 4)	3
0.1	(1.1, 1.21)	2.1
0.01	(1.01, 1.0201)	2.01
0.001	(1.001, 1.002001)	2.001

The slopes get closer and closer to 2 .

2. Instantaneous rate of change: letting h shrink

The table suggests a number the slopes are heading toward. To capture an *instantaneous* rate we let h approach 0 without ever setting $h = 0$ (that would give $0/0$). The limit of the average rates is the instantaneous rate of change.



Common mistake

You cannot plug $h = 0$ into the difference quotient directly. First simplify algebraically (expand, factor, cancel h), and only then take the limit.

3. The derivative as a limit



Definition: derivative

The **derivative of f at a** is

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h},$$

when this limit exists. If it exists, f is **differentiable at a** . The **derivative function** is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}. \text{ Other notations: } \frac{dy}{dx} \text{ and } \frac{d}{dx} f(x).$$

The number $f'(a)$ has two meanings: the **slope of the tangent line** at $(a, f(a))$, and the **instantaneous rate of change** of f at a , measured in units of f per unit of x (for example feet per second).

E Example 2: derivative from the definition

Let $f(x) = 3x^2 - 5x$. Then

$$\frac{f(x+h) - f(x)}{h} = \frac{3(x^2 + 2xh + h^2) - 5x - 5h - 3x^2 + 5x}{h} = \frac{6xh + 3h^2 - 5h}{h}$$

As $h \rightarrow 0$, this tends to $6x - 5$. So $f'(x) = 6x - 5$ and, for instance, $f'(2) = 7$.

E Example 3: a square root

Let $f(x) = \sqrt{x}$ at $a = 4$. Multiply by the conjugate:

$$\frac{\sqrt{4+h} - 2}{h} = \frac{(4+h) - 4}{h(\sqrt{4+h} + 2)} = \frac{1}{\sqrt{4+h} + 2} \rightarrow \frac{1}{4}.$$

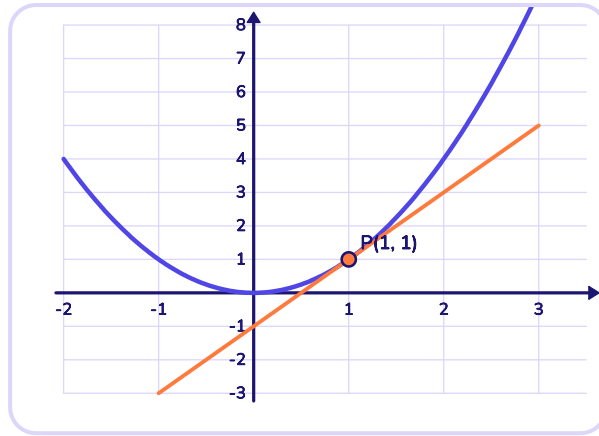
So $f'(4) = \frac{1}{4}$: near $x = 4$, the output grows by about a quarter of the input change.

4. The tangent line equation

P Tangent line

If f is differentiable at a , the tangent line to the graph at $(a, f(a))$ has slope $f'(a)$ and equation

$$y = f(a) + f'(a)(x - a).$$



M Method: find a tangent line

1. Compute the point: $f(a)$.
2. Compute the slope: $f'(a)$.
3. Substitute into $y = f(a) + f'(a)(x - a)$ and simplify.

E Example 4: tangent line

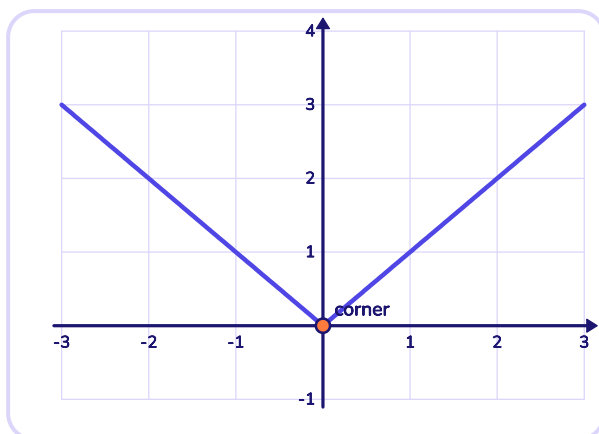
Find the tangent to $f(x) = x^3 - 3x$ at $a = 2$. We have $f(2) = 8 - 6 = 2$. Using the rules of Part 6 and 7, $f'(x) = 3x^2 - 3$, so $f'(2) = 9$. The tangent is $y = 2 + 9(x - 2) = 9x - 16$.

5. Differentiability and continuity

P Theorem

If f is differentiable at a , then f is continuous at a .

Why. For $h \neq 0$, write $f(a+h) - f(a) = \frac{f(a+h) - f(a)}{h} \cdot h$. As $h \rightarrow 0$ the first factor tends to $f'(a)$ and the second to 0, so $f(a+h) - f(a) \rightarrow 0$. That is exactly $\lim_{x \rightarrow a} f(x) = f(a)$.



The converse is false

The function $f(x) = |x|$ is continuous at 0, but not differentiable there. For $h > 0$ the quotient is $\frac{|h|}{h} = 1$; for $h < 0$ it is -1 . The two one-sided limits disagree, so $f'(0)$ does not exist. A function typically fails to be differentiable at a **corner**, at a **vertical tangent** (the quotient tends to infinity), or at a **discontinuity**.

6. The power rule



Power rule

For any real number n (with x in the domain where the expressions make sense),

$$\frac{d}{dx}(x^n) = n x^{n-1}.$$

Special cases: $\frac{d}{dx}(x) = 1$ and, for any constant c , $\frac{d}{dx}(c) = 0$.

For a positive integer n , the binomial expansion gives

$(x + h)^n = x^n + nx^{n-1}h + (\text{terms with } h^2)$, so the quotient is $nx^{n-1} + (\text{terms with } h)$, which tends to nx^{n-1} .

E Example 5: rewrite, then apply the rule

$\frac{1}{x^3} = x^{-3}$, so its derivative is $-3x^{-4} = -\frac{3}{x^4}$. Also $\sqrt{x} = x^{1/2}$, so its derivative is $\frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$, which matches Example 3: at $x = 4$ it gives $\frac{1}{4}$.

! Zyro's tip

On my home planet we say: "Radical or fraction? Make it a power first." Rewriting roots and reciprocals as powers of x lets you use one rule for everything.

7. Sum and constant multiple rules

P Linearity of the derivative

If f and g are differentiable and c is a constant, then

$$(f + g)' = f' + g', \quad (f - g)' = f' - g', \quad (cf)' = cf'.$$

E**Example 6: a polynomial**

For $f(x) = 4x^5 - 7x^3 + 2x - 9$:

$$f'(x) = 4 \cdot 5x^4 - 7 \cdot 3x^2 + 2 \cdot 1 - 0 = 20x^4 - 21x^2 + 2.$$

Then $f'(1) = 20 - 21 + 2 = 1$.

**Careful**

The sum rule does *not* extend to products: $(f \cdot g)'$ is not $f' \cdot g'$. For example $(x \cdot x)' = (x^2)' = 2x$, not $1 \cdot 1$.

Key takeaways

- The difference quotient $\frac{f(a+h) - f(a)}{h}$ is the average rate of change and the slope of a secant line.
- $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ is the instantaneous rate of change and the slope of the tangent line.
- Tangent line at a : $y = f(a) + f'(a)(x - a)$.
- Differentiable at a implies continuous at a ; the converse fails (corners, vertical tangents).
- Power rule: $(x^n)' = nx^{n-1}$; constants have derivative **0**.
- Sum and constant multiple rules let you differentiate polynomials term by term.



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