

Applications of Integration: math lesson, College – download the PDF

You already know that a definite integral adds up infinitely many tiny pieces. That single idea is powerful: slice a region, a solid, a curve, or a motion into thin pieces, estimate each piece, then integrate. In this chapter you will use it to find areas between curves, volumes of solids, lengths of curves, average values, work done by a force, and the solutions of simple differential equations.

1. Area between two curves

The area under one curve is $\int_a^b f(x) dx$. To find the area *between* two curves, subtract the lower curve from the upper curve before integrating.

P

Area between curves

If $f(x) \geq g(x)$ on $[a, b]$, the area between the graphs is

$$A = \int_a^b (f(x) - g(x)) dx.$$

If the roles are easier to describe with y , use $A = \int_c^d (x_{\text{right}}(y) - x_{\text{left}}(y)) dy$.

M**Method**

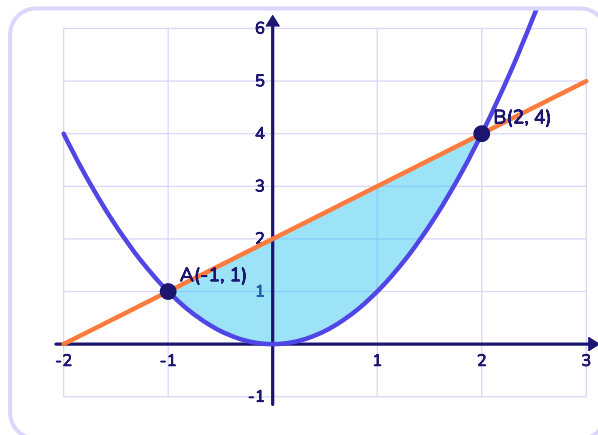
1. Sketch the curves and find where they meet by solving $f(x) = g(x)$.
2. On each interval between meeting points, decide which curve is on top.
3. Integrate top minus bottom. If the curves cross, split the integral at the crossing.

E**Example 1**

Find the area enclosed by $y = x + 2$ and $y = x^2$.

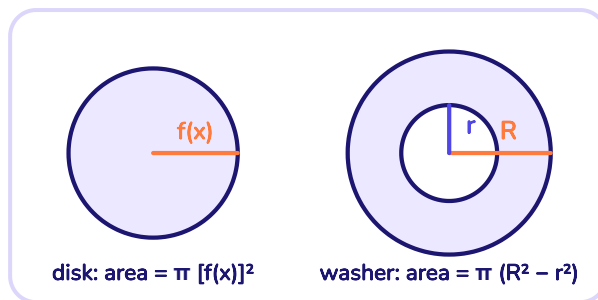
Solve $x^2 = x + 2$: $x^2 - x - 2 = (x - 2)(x + 1) = 0$, so $x = -1$ or $x = 2$. On $(-1, 2)$ the line is above the parabola, so

$$A = \int_{-1}^2 (x + 2 - x^2) dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^2 = \frac{10}{3} - \left(-\frac{7}{6} \right) = \frac{9}{2}.$$



2. Volumes by disks and washers

Spin a region around a line and it sweeps out a solid. Slice the solid perpendicular to the axis and every slice is either a disk or a washer (a disk with a hole).



P Disk and washer formulas

Rotating the region under $y = f(x)$ on $[a, b]$ about the x-axis gives

$$V = \pi \int_a^b [f(x)]^2 dx.$$

If the region lies between an outer curve $R(x)$ and an inner curve $r(x)$, then

$$V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2) dx.$$

E Example 2 (disks)

Rotate the region under $y = \sqrt{x}$ on $[0, 4]$ about the x-axis:

$$V = \pi \int_0^4 x dx = \pi \cdot 8 = 8\pi \approx 25.13 \text{ cubic units.}$$

E Example 3 (washers)

The region between $y = x$ and $y = x^2$ on $[0, 1]$ is rotated about the x-axis. The outer radius is x and the inner radius is x^2 :

$$V = \pi \int_0^1 (x^2 - x^4) dx = \pi \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{2\pi}{15} \approx 0.419.$$



Common mistake

Square the whole radius, not each term: $\pi(R^2 - r^2)$ is correct, but $\pi(R - r)^2$ is not.

When the axis is a line such as $y = -1$, the radii are *distances to that line*.

3. Volumes by cylindrical shells

Instead of slicing perpendicular to the axis, slice parallel to it. A thin vertical strip rotated about a vertical axis becomes a hollow cylinder, a shell, whose unrolled surface is a thin rectangle.



Shell formula

For a region between $x = a$ and $x = b$ rotated about a vertical axis,

$$V = 2\pi \int_a^b (\text{radius})(\text{height}) dx,$$

where the radius is the distance from the strip to the axis and the height is the length of the strip. For the y-axis, the radius is simply x .

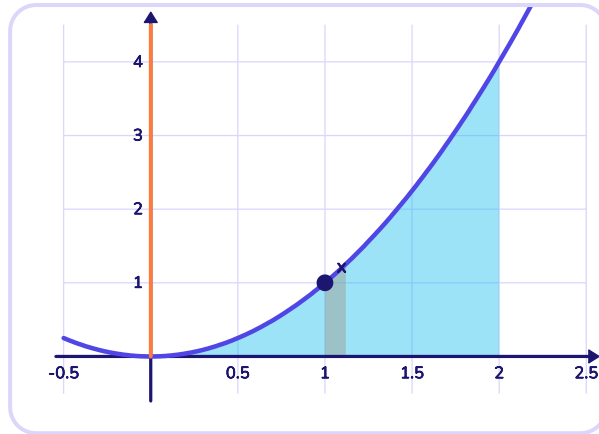


Example 4

Rotate the region under $y = x^2$ on $[0, 2]$ about the y-axis. Radius x , height x^2 :

$$V = 2\pi \int_0^2 x \cdot x^2 dx = 2\pi \left[\frac{x^4}{4} \right]_0^2 = 8\pi.$$

Check with disks in y : a cylinder of radius 2 and height 4 has volume 16π ; remove the paraboloid $\pi \int_0^4 y dy = 8\pi$ to get 8π again.



Which method?

Zyro the explorer says: pick the method that avoids rewriting the curve. If the slices are perpendicular to the axis, use disks or washers; if they are parallel to it, use shells.

4. Arc length

Zoom in on a smooth curve and it looks like a tiny straight segment of length

$\sqrt{dx^2 + dy^2} = \sqrt{1 + (dy/dx)^2} dx$ (the Pythagorean theorem). Adding those pieces gives the length.



Arc length formula

If f' is continuous on $[a, b]$, the length of $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx.$$

E Example 5

Find the length of $y = \frac{2}{3}x^{3/2}$ for $0 \leq x \leq 3$. Here $f'(x) = x^{1/2}$, so $1 + (f')^2 = 1 + x$ and

$$L = \int_0^3 \sqrt{1+x} \, dx = \frac{2}{3} \left[(1+x)^{3/2} \right]_0^3 = \frac{2}{3} (8 - 1) = \frac{14}{3} \approx 4.67.$$



Common mistake

The square root rarely simplifies by luck. If it does not, either use a substitution or evaluate the integral numerically.

5. Average value of a function



Average value

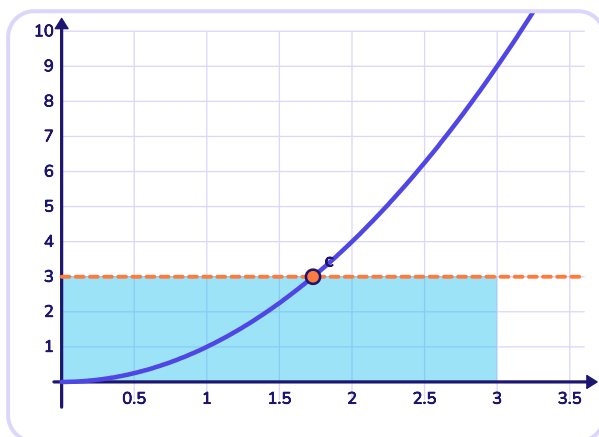
The average value of f on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) \, dx.$$

If f is continuous, the Mean Value Theorem for Integrals guarantees a point c in $[a, b]$ with $f(c) = f_{\text{avg}}$.

E Example 6

For $f(x) = x^2$ on $[0, 3]$: $f_{\text{avg}} = \frac{1}{3} \int_0^3 x^2 dx = \frac{1}{3} \cdot 9 = 3$. The value 3 is reached where $c^2 = 3$, that is $c = \sqrt{3} \approx 1.73$. The rectangle of height 3 has the same area as the region under the curve.



6. Work and force

Work is force times distance. When the force changes with position, add up the work done over tiny steps.

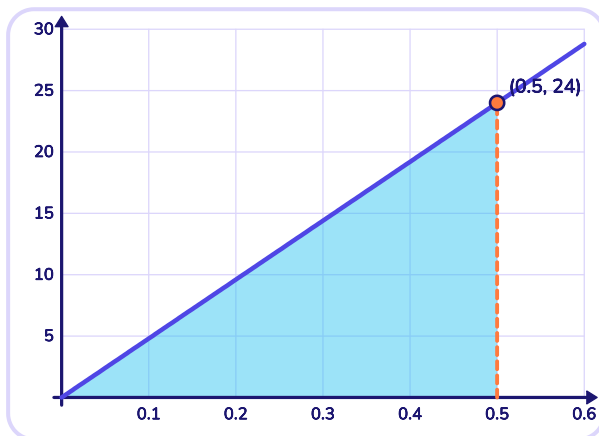
P Work done by a variable force

If a force $F(x)$ acts along a line from $x = a$ to $x = b$, then $W = \int_a^b F(x) dx$. Units: foot-pounds (ft·lb) or joules ($\text{J} = \text{N} \cdot \text{m}$); $1 \text{ ft} \cdot \text{lb} \approx 1.356 \text{ J}$. Hooke's law for a spring is $F = kx$, where x is the distance stretched from natural length.

E Example 7 (spring)

A spring needs 12 lb to stay stretched 3 in (0.25 ft), so $k = 12/0.25 = 48$ lb/ft.

Stretching it 0.5 ft (6 in) takes $W = \int_0^{0.5} 48x \, dx = 24x^2 \Big|_0^{0.5} = 6 \text{ ft}\cdot\text{lb} \approx 8.1 \text{ J}$, which is the shaded triangle below.



E Example 8 (hanging cable)

A 40-ft cable weighing 0.5 lb per foot hangs from a roof. A piece of length dy at depth y must be lifted y feet and weighs $0.5 \, dy$ pounds, so

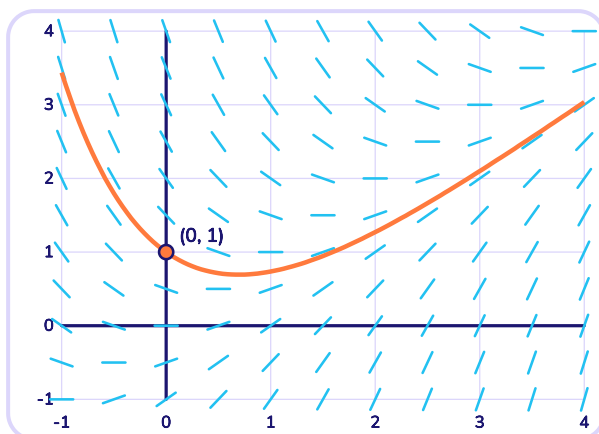
$$W = \int_0^{40} 0.5 \, y \, dy = 0.25 \cdot 40^2 = 400 \text{ ft}\cdot\text{lb}.$$

7. Differential equations: the basics

D Differential equation

A differential equation (DE) relates an unknown function to its derivatives, such as $y' = 2y$. Its *order* is the highest derivative that appears. The *general solution* has an arbitrary constant C ; an *initial-value problem* adds a condition such as $y(0) = 1$ that fixes C .

The equation $y' = F(x, y)$ tells you the slope of the solution at every point. Drawing a tiny segment with that slope at many points gives a *slope field*, and a solution curve follows the segments.



E Example 9 (checking a solution)

Show that $y = x - 1 + 2e^{-x}$ solves $y' = x - y$ with $y(0) = 1$. Differentiate: $y' = 1 - 2e^{-x}$. Also $x - y = x - (x - 1 + 2e^{-x}) = 1 - 2e^{-x}$, so the two sides match. And $y(0) = -1 + 2 = 1$.

I 8. Separable equations

M Method for $\frac{dy}{dx} = h(x) g(y)$

1. Separate the variables: $\frac{dy}{g(y)} = h(x) dx$.
2. Integrate both sides, adding one constant C .
3. Solve for y if possible, then use the initial condition to find C .

E**Example 10**

Solve $y' = 3x^2y$, $y(0) = 2$. Separate: $\frac{dy}{y} = 3x^2 dx$, so $\ln |y| = x^3 + C$ and $y = Ae^{x^3}$. From $y(0) = 2$ we get $A = 2$, so $y = 2e^{x^3}$.

E**Example 11 (population growth)**

A culture of 2,000 bacteria grows at a rate proportional to its size: $\frac{dP}{dt} = 0.05P$, $P(0) = 2000$ with t in hours. Then $P = 2000e^{0.05t}$ and $P(10) = 2000e^{0.5} \approx 3,297$.

Key takeaways

- Area between curves: integrate top minus bottom, splitting wherever the curves cross.
- Disks and washers: $\pi \int (R^2 - r^2) dx$; shells: $2\pi \int (\text{radius})(\text{height}) dx$.
- Arc length: $\int \sqrt{1 + (f')^2} dx$.
- Average value: $\frac{1}{b-a} \int_a^b f(x) dx$.
- Work: $\int F(x) dx$; for a spring $F = kx$ and $W = \frac{1}{2}kx^2$.
- Separable equations: separate, integrate both sides, then use the initial condition.

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