

Applications of Derivatives: math lesson, College – download the PDF

Derivatives are not just a calculation drill: they are the tool that tells you **where a function rises, where it falls, and how fast things change**. In this chapter you will use f' and f'' to sketch curves, to find the best possible value in a real situation, to track quantities that change together, and to compute tricky limits.

1. Increasing and decreasing functions

P Sign of the derivative

Let f be differentiable on an interval I . If $f'(x) > 0$ for every x in I , then f is increasing on I . If $f'(x) < 0$ for every x in I , then f is decreasing on I .

The derivative is the slope of the tangent line: a positive slope means the graph climbs from left to right, a negative slope means it descends. To find the intervals, solve $f'(x) = 0$, place the solutions on a number line, and test one value in each piece.

E Example 1

Let $f(x) = x^3 - 6x^2 + 9x + 1$. Then $f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$, which is zero at $x = 1$ and $x = 3$.

Interval	$x < 1$	$1 < x < 3$	$x > 3$
$f'(x) = 3(x - 1)(x - 3)$	+	-	+
Behavior of f	increasing	decreasing	increasing



So f increases on $(-\infty, 1)$ and $(3, \infty)$ and decreases on $(1, 3)$.

2. Critical points and extrema

D Critical point

A number c in the domain of f is a **critical point** if $f'(c) = 0$ or $f'(c)$ does not exist. A **local maximum** (or minimum) at c means $f(c)$ is the largest (or smallest) value of f near c .

P First derivative test

At a critical point c : if f' changes from $+$ to $-$, then $f(c)$ is a local maximum; if it changes from $-$ to $+$, then $f(c)$ is a local minimum; if the sign does not change, there is no extremum at c .

In Example 1, f' goes from $+$ to $-$ at $x = 1$, so $f(1) = 5$ is a local maximum; it goes from $-$ to $+$ at $x = 3$, so $f(3) = 1$ is a local minimum. On a closed interval $[a, b]$, the absolute extrema occur at critical points or at the endpoints, so compare all of those values.



Common mistake

$f'(c) = 0$ does **not** guarantee an extremum. For $f(x) = x^3$ we have $f'(0) = 0$, yet the function keeps increasing through 0.

3. Concavity and inflection points



Concavity

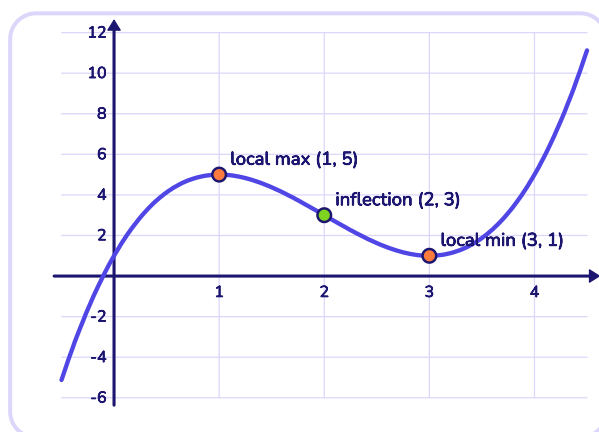
If $f''(x) > 0$ on an interval, the graph is **concave up** (it bends like a cup). If $f''(x) < 0$, it is **concave down**. An **inflection point** is a point of the graph where the concavity changes.

For Example 1, $f''(x) = 6x - 12$, which is negative for $x < 2$ and positive for $x > 2$. The graph is concave down on $(-\infty, 2)$, concave up on $(2, \infty)$, and $(2, f(2)) = (2, 3)$ is an inflection point. The **second derivative test** also helps: if $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a local minimum; if $f''(c) < 0$, a local maximum. Check: $f''(1) = -6 < 0$ (maximum) and $f''(3) = 6 > 0$ (minimum).

4. Curve sketching

M Method

1. Find the domain and the intercepts when they are easy.
2. Compute f' , solve $f' = 0$, and build the sign chart.
3. Compute f'' and find the concavity and inflection points.
4. Evaluate f at every special point, then connect the points with a smooth curve that respects the signs.



The figure above is exactly this method applied to Example 1: a hill at $(1, 5)$, a valley at $(3, 1)$, and the bend at $(2, 3)$, halfway between them.

5. The Mean Value Theorem

P Mean Value Theorem (MVT)

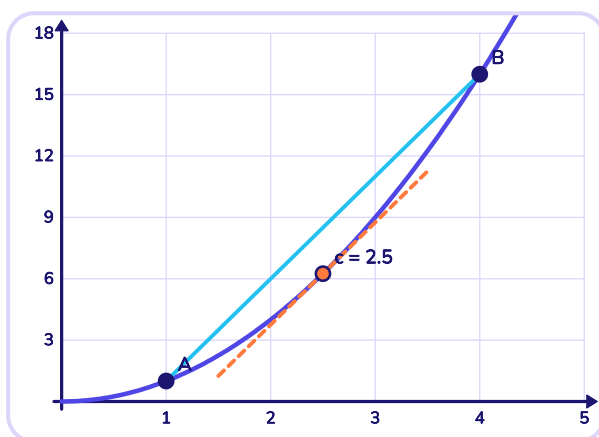
If f is continuous on $[a, b]$ and differentiable on (a, b) , then there is at least one number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In words: somewhere between a and b , the instantaneous rate of change equals the average rate of change. Geometrically, some tangent line is parallel to the secant line.

E Example 2

For $f(x) = x^2$ on $[1, 4]$, the average rate is $\frac{16 - 1}{4 - 1} = 5$. Solve $f'(c) = 2c = 5$, so $c = 2.5$, which lies in $(1, 4)$.



I 6. Optimization problems

M Method

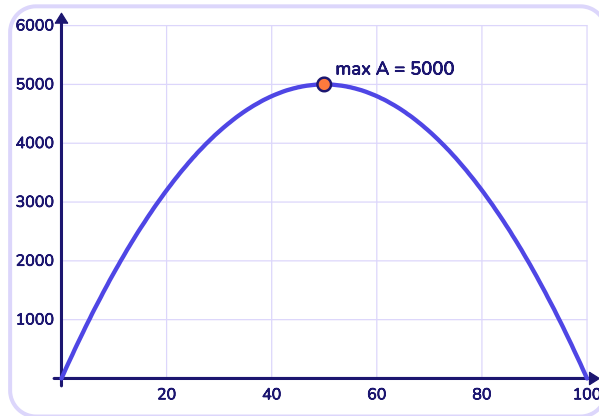
1. Draw a picture and name the variables.
2. Write the quantity to optimize, then use the constraint to reduce it to **one** variable.
3. State the allowed interval, find the critical points, and compare the values (including endpoints).
4. Answer the question with units.

E**Example 3**

You have 200 ft of fencing for a rectangular garden that uses a long wall as one side, so only three sides need fencing. Let x be the length of each of the two sides perpendicular to the wall. The third side is $200 - 2x$, so

$$A(x) = x(200 - 2x) = 200x - 2x^2 \text{ for } 0 \leq x \leq 100.$$

$A'(x) = 200 - 4x = 0$ gives $x = 50$. Since $A(0) = A(100) = 0$ and $A(50) = 50 \cdot 100 = 5000$, the maximum area is 5000 ft^2 (about 464.5 m^2), with dimensions $50 \text{ ft} \times 100 \text{ ft}$.

**7. Related rates**

When several quantities change with time and are linked by an equation, differentiate that equation with respect to t (using the chain rule), then substitute the known values **after** differentiating.

E

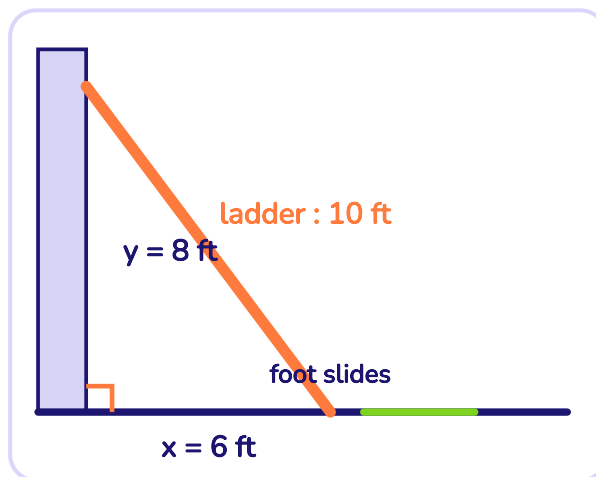
Example 4

A 10 ft ladder leans against a wall. Its foot is $x = 6$ ft from the wall and slides away at $\frac{dx}{dt} = 2$ ft/s. How fast does the top move down?

The ladder, wall, and floor form a right triangle: $x^2 + y^2 = 100$. At that moment

$y = \sqrt{100 - 36} = 8$. Differentiating: $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$, so

$\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt} = -\frac{6}{8} \cdot 2 = -1.5$ ft/s. The top slides down at 1.5 ft/s (about 0.46 m/s).



Zyro's tip

Never plug the numbers in before you differentiate: a quantity that is changing must stay a variable until the derivative is taken!

8. L'Hôpital's rule

P L'Hôpital's rule

Suppose $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ has the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ (as $x \rightarrow a$, or as $x \rightarrow \pm\infty$). If $g'(x) \neq 0$ near a and $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

E Example 5

$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$ is of the form $\frac{0}{0}$. Apply the rule: $\frac{\sin x}{2x}$, still $\frac{0}{0}$; apply again: $\frac{\cos x}{2} \rightarrow \frac{1}{2}$. Also $\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0$.



Common mistake

Differentiate the numerator and the denominator **separately**. Do not use the quotient rule, and do not use L'Hôpital's rule when the limit is not indeterminate.

9. Linear approximation

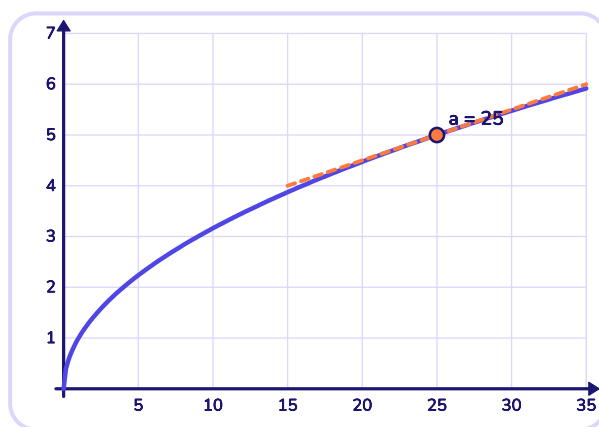
D Linearization

Near $x = a$, a differentiable function is well approximated by its tangent line:

$$f(x) \approx L(x) = f(a) + f'(a)(x - a).$$

E Example 6

Estimate $\sqrt{26}$. Take $f(x) = \sqrt{x}$ and $a = 25$: $f(25) = 5$ and $f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{10}$. So $L(26) = 5 + \frac{1}{10} = 5.1$. The calculator gives **5.0990**, so the error is about **0.001**.



Key takeaways

- $f' > 0$ means increasing, $f' < 0$ means decreasing; extrema occur at critical points where the sign of f' changes.
- $f'' > 0$ means concave up, $f'' < 0$ means concave down; an inflection point is where concavity changes.
- MVT: some c has $f'(c) = \frac{f(b) - f(a)}{b - a}$.
- Optimization: one variable, an interval, critical points, and endpoints.
- Related rates: differentiate with respect to t first, substitute after.
- L'Hôpital's rule only for $\frac{0}{0}$ and $\frac{\infty}{\infty}$; linearization: $f(x) \approx f(a) + f'(a)(x - a)$.



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